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Mathematics. — *"Intuitionistischer Beweis des Fundamentalsatzes der Algebra".* By Prof. L. E. J. Brouwer and B. de Loor.

(Communicated at the meeting of February 23, 1924).

Sei $\psi(x) \equiv x^n + a_1 x^{n-1} + \dots a_n = 0$ eine algebraische Gleichung mit rationalkomplexen Koeffizienten. Hiermit meinen wir, dass jedes $a_\nu = b_\nu + ic_\nu$ ist, wo b_ν und c_ν reelle rationale Zahlen vorstellen. Diese Gleichung lässt sich derweise in eine endliche Anzahl von algebraischen Gleichungen zerlegen, dass jede dieser Faktorgleichungen ebenfalls rationalkomplexe Koeffizienten besitzt, während überdies ihr linkes Glied und dessen Derivierte relativ prim sind. Sei $\varphi(x) \equiv x^p + e_1 x^{p-1} + \dots e_p = 0$ eine dieser Faktorgleichungen. Alsdann können zwei solche ganze rationale Funktionen $h(x)$ und $g(x)$ mit rationalkomplexen Koeffizienten bestimmt werden, dass $h(x)\varphi(x) + g(x)\varphi'(x) \equiv 1$ ist, so dass man zwei solche positive Grössen a und b angeben kann, dass $|\varphi'(x)| > b$ gilt für $|\varphi(x)| < a$. Wir stellen uns zum Ziel, von $\varphi(x) = 0$ eine Wurzel zu bestimmen und werden dabei voraussetzen, dass $|\varphi(0)| > 0$ ist (worin keine Beschränkung gelegen ist).

Sei $|\varphi| - |\varphi - x^p| > k > 0$ für $|x| \geq r$ und $|\varphi'| < q$ für $|x| \leq r$. Sei ε eine beliebige positive Grösse. Im Innengebiete des in der x-Ebene mit dem Radius r um den Nullpunkt geschlagenen Kreises $C(r)$ bestimmen wir eine solche endliche Anzahl von Punkten $P_1, P_2, \dots P_m$, dass jeder beliebige innerhalb oder auf $C(r)$ gelegene Punkt einen Abstand $< \frac{\varepsilon}{q}$ von der Menge der P_ν besitzt. Alsdann besitzt in der φ-Ebene jeder Bildpunkt eines innerhalb oder auf $C(r)$ gelegenen Punktes der x-Ebene einen Abstand $< \varepsilon$ von der Menge der Bildpunkte Q_ν der P_ν. Nehmen wir einen Augenblick an, dass diese Bildpunkte Q_ν alle einen Modulus $> \varepsilon$ besässen. Dann könnte eine solche positive Grösse η bestimmt werden, dass von allen in der x-Ebene mit Radien $\varrho \leq r$ um den Nullpunkt geschlagenen Kreisen $C(\varrho)$ die Bilder in der φ-Ebene in einer Entfernung $\geq \eta$ vom Nullpunkte blieben. Dies aber ist ungereimt, denn das Bild von $C(\varrho)$ umkreist bei einem vollen Umlaufe den Nullpunkt der φ-Ebene p-mal für $\varrho = r$ und 0-mal, wenn ϱ hinreichend klein gewählt wird. Mithin kann in der φ-Ebene ein Punkt Q_s, dessen Modulus $\leq \varepsilon$ ist, angegeben werden. Sei $x = x_1$ im entsprechenden Punkte P_2 der x-Ebene.

Wir setzen nun voraus, dass die im vorigen Absatze auftretende positive Grösse ε dergestalt gewählt ist, dass erstens ε sowohl $< k$ wie $< a$ ist, und zweitens innerhalb eines um einen innerhalb oder auf $C(r)$ gelegenen Punkt, wo $|\varphi'| > b$ ist, mit dem Radius $\frac{\varepsilon}{b}$ geschlagenen Kreises die Ungleichung

$$\frac{|\varphi_2 - \varphi_1 - (x_2 - x_1)\varphi'_1|}{|(x_2 - x_1)\varphi'_1|} < \frac{1}{4}$$

in Kraft ist. Setzen wir unter dieser Voraussetzung $x_2 = x_1 - \frac{\varphi(x_1)}{\varphi'(x_1)}$, $x_3 = x_2 - \frac{\varphi(x_2)}{\varphi'(x_2)}$, usw., so wird der Reihe nach $|\varphi(x_2)| < \frac{1}{4}\varepsilon$, $|\varphi(x_3)| < \frac{1}{16}\varepsilon$, usw., während die unendliche Reihe $x_1, x_2, x_3, \ldots.$ gegen einen bestimmbaren einzigen Grenzwert x' konvergiert. Dann aber gilt notwendig $\varphi(x') = 0$, und wir haben $\varphi(x) \equiv (x - x')\chi(x)$, wo $\chi(x) \equiv x^{p-1} + h_1 x^{p-2} + \ldots\ldots h_{p-1}$.

Für die Gleichung $\chi(x) = 0$ (die im allgemeinen nicht mehr rationalkomplexe Koeffizienten besitzt) können wir zwei solche positive Grössen c und $\alpha < \frac{1}{2} b$ bestimmen, dass aus $|\chi| < \alpha$ folgt $|\varphi| < a$ und $|x - x'| < c$. Dann aber folgt wegen $\varphi' \equiv (x - x')\chi' + \chi$ aus $|\chi| < \alpha$ weiter $|(x - x')\chi'| > \frac{1}{2} b$, mithin, wenn wir $\frac{b}{2c} = \beta$ setzen, $|\chi'| > \beta$. Weil mithin α und β für χ dieselbe Rolle spielen, wie a und b für φ, so können wir in derselben Weise, wie wir für $\varphi = 0$ eine Wurzel x' bestimmt haben, für $\chi = 0$ eine Wurzel x'' berechnen. Indem wir in dieser Weise fortfahren, zerlegen wir zunächst $\varphi(x)$ in p Faktoren und sodann $\psi(x)$ in n Faktoren der Form $x - x^{(\nu)}$.

Sei nun $\psi_\nu(x) \equiv x^n + a_{\nu 1} x^{n-1} + \ldots . a_{\nu n}$ für jedes ganze positive ν, während alle $a_{\nu r}$ rationalkomplexe Zahlen sind und die unendliche Funktionenfolge $\psi_1(x)$, $\psi_2(x), \ldots.$ gegen eine einzige ganze rationale Funktion $f(x) \equiv x^n + b_1 x^{n-1} + \ldots . b_n$ konvergiert. Wir werden zeigen, dass auch von der Gleichung $f(x) = 0$ eine Wurzel ermittelt werden kann, womit der Fundamentalsatz der Algebra bewiesen sein wird.

Wenn wir die Wurzeln von $\psi_\nu(x) = 0$ mit $x'_\nu, \ldots x^{(n)}_\nu$ bezeichnen, dann können wir für ein beliebiges positives ε_m ein solches ν_m bestimmen, dass für ein beliebiges $x^{(\rho)}_{\nu_m}$ und für eine beliebige natürliche Zahl λ die Ungleichung

$$\left|\psi_{\nu_m+\lambda}(x^{(\rho)}_{\nu_m})\right| < \varepsilon^n_m$$

gilt, so dass für eine irgendwie ausgewählte Wurzel $x^0_{\nu_m}$ von $\psi_{\nu_m}(x)=0$ und für eine beliebige natürliche Zahl λ das Produkt

$$|x'_{\nu_m+\lambda} - x^0_{\nu_m}| \ldots . |x^{(n)}_{\nu_m+\lambda} - x^0_{\nu_m}|$$

$< \varepsilon^n_m$ und somit wenigstens einer seiner Faktoren $< \varepsilon_m$ ist.

Es konvergiere $\varepsilon_1 + \varepsilon_2 + \ldots .$ gegen ε. Wir bestimmen in der im vorigen Absatz angegebenen Weise zu jedem ε_m ein passendes ν_m, wobei wir dafür sorgen, dass stets $\nu_{m+1} > \nu_m$ ist. Im Anschluss daran wählen wir für $x^0_{\nu_1}$ eine beliebige Wurzel von $\psi_{\nu_1}(x) = 0$ und sodann nach dem vorigen Absatz für $x^0_{\nu_{m+1}}$ jedesmal eine derartige Wurzel von $\psi_{\nu_{m+1}}(x) = 0$, dass $|x^0_{\nu_{m+1}} - x^0_{\nu_m}| < \varepsilon_m$ ist. Alsdann konvergiert die unendliche Folge $x^0_{\nu_1}, x^0_{\nu_2}, \ldots .$ gegen einen einzigen Limeswert x^0_ω und dieser Limeswert ist eine Wurzel von $f(x) = 0$.

Mathematics. — "*Beweis, dass jede volle Funktion gleichmässig stetig ist*". By Prof. L. E. J. Brouwer.

(Communicated at the meeting of March 29, 1924).

§ 1.

Sei M eine beliebige Menge, μ die ihr zu Grunde liegende zählbare (übrigens auch abzählbar unendliche) Menge der endlichen (gehemmten und ungehemmten) Wahlfolgen $F_{sn_1 \ldots n_r}$, wo s und die n_ν natürliche Zahlen vorstellen, und sei jedem Elemente von M eine natürliche Zahl β zugeordnet. Alsdann ist in μ eine solche abtrennbare zählbare Teilmenge μ_1 von ungehemmten endlichen Wahlfolgen ausgezeichnet, dass einem beliebigen Elemente von μ_1 für alle aus ihm hervorgehenden Elemente von M eine natürliche Zahl β zugeordnet ist, während weiter eine Beweisführung h vorliegt, mittels welcher sich für ein beliebiges ungehemmtes Element von μ herausstellt, dass jede aus ihm hervorgehende ungehemmte unendliche Wahlfolge einen zu μ_1 gehörigen Abschnitt besitzt. (Ein ungehemmtes Element von μ soll nämlich dann und nur dann zu μ_1 gerechnet werden, wenn bei ihm — aber bei keinem seiner echten Abschnitte — nach dem *Algorithmus* des Zuordnungsgesetzes die Entscheidung hinsichtlich β *nicht* bis auf weitere Wahlen aufgeschoben wird; dabei ist es selbstverständlich keineswegs ausgeschlossen, dass man hinterher auch weder zu μ_1 gehörige noch einen zu μ_1 gehörigen Abschnitt besitzende Elemente von μ angeben kann mit der Eigenschaft, dass allen aus einem solchen Elemente von μ hervorgehenden Elementen von M dieselbe natürliche Zahl zugeordnet ist).

Nennen wir ein Element von μ *versichert,* wenn es entweder gehemmt ist oder einen zu μ_1 gehörigen (echten oder nicht echten Abschnitt besitzt, so ist μ in eine zählbare Menge τ von versicherten und eine zählbare Menge σ von nicht versicherten endlichen Wahlfolgen zerlegt, und die Beweisführung h zeigt *für ein beliebiges Element von* σ, dass es *versicherbar* ist, d. h. dass jede aus ihm hervorgehende für M ungehemmte unendliche Wahlfolge einen zu μ_1 gehörigen Abschnitt besitzt. Sei $h_{sn_1 \ldots n_r}$ die Spezialisierung von h

welche die Versicherbarkeit des Elementes $F_{sn_1\ldots n_r}$ von σ herleitet, so beruht diese Beweisführung ausschliesslich auf den zwischen den Elementen von μ bestehenden Beziehungen b. Diese Beziehungen aber sind alle in *Elementarbeziehungen* e zerlegbar von der Art, wie sie zwischen je einem *einzigen* Elementepaar $F_{mm_1\ldots m_g}$ und $F_{mm_1\ldots m_g m_{g+1}}$ bestehen, von denen das eine eine unmittelbare Verlängerung des anderen ist. Weil nun eine beliebige Beweisführung, wenn die in derselben benutzten Beziehungen in Grundbeziehungen zerlegbar sind, sich immer (wenn auch auf Kosten der Kürze) derart "kanonisieren" lässt, dass in ihrer kanonisierten Form nur noch die Grundbeziehungen benutzt werden, so kann bei der kanonisierten Form $k_{sn_1\ldots n_r}$ der Beweisführung $h_{sn_1\ldots n_r}$ die Versicherbarkeit von $F_{sn_1\ldots n_r}$ in letzter Instanz ausschliesslich aus den Elementarbeziehungen e gefolgert werden, welche $F_{sn_1\ldots n_r}$ mit $F_{sn_1\ldots n_{r-1}}$ und mit den $F_{sn_1\ldots n_r\nu}$ verbinden. Zum Schlussglied von $k_{sn_1\ldots n_r}$ braucht man also die vorherige Feststellung der Versicherbarkeit entweder von $F_{sn_1\ldots n_{r-1}}$ oder von *allen* $F_{sn_1\ldots n_r\nu}$.

Nennen wir einen Elementarschluss, der die Versicherbarkeit eines $F_{mm_1\ldots m_g}$ aus derjenigen von $F_{mm_1\ldots m_{g-1}}$ folgert, einen *ζ-Schluss*, einen Elementarschluss, der die Versicherbarkeit eines $F_{mm_1\ldots m_g}$ aus derjenigen von *allen* $F_{mm_1\ldots m_g\nu}$ folgert, einen *$\digamma$-Schluss*, und bezeichnen wir die wohlgeordnete Spezies von Elementen von σ, von denen bei der Beweisführung $k_{sn_1\ldots n_r}$ der Reihe nach die Versicherbarkeit festgestellt wird, mit $f_{sn_1\ldots n_r}$, so kann vom *ersten* Elemente von $f_{sn_1\ldots n_r}$ die Versicherbarkeit unmöglich mittels eines ζ-Schlusses hergeleitet werden, so dass sie also mittels eines $\digamma$-Schlusses gefolgert werden muss. Auf Grund transfiniter Induktion längs $f_{sn_1\ldots n_r}$ ersehen wir weiter, dass *in jedem Stadium der Beweisführung* $k_{sn_1\ldots n_r}$ von allen schon als versicherbar erkannten Elementen von σ die Verlängerungen ebenfalls schon als versicherbar erkannt worden sind, so dass bei $k_{sn_1\ldots n_r}$ von *jedem* Elemente von $f_{sn_1\ldots n_r}$ die Versicherbarkeit mittels eines $\digamma$-Schlusses gefolgert wird.

Wenn wir nun (in Uebereinstimmung mit den zwischen den entsprechenden Teilmengen von M bestehenden Vereinigungsbeziehungen) jedesmal, wenn bei $k_{sn_1\ldots n_r}$ von einem Elemente $F_{mm_1\ldots m_g}$ von σ die Versicherbarkeit festgestellt wird, dieses Element als nach dem Index ν geordnete Summe der $F_{mm_1\ldots m_g\nu}$ (und dabei jedes gehemmte

$F_{mm_1\ldots m_{g^\nu}}$ als elementlose Urspezies) auffassen, so ergibt sich auf Grund transfiniter Induktion längs $f_{sn_1\ldots n_r}$, dass durch diese transfinite Reihe von Summenbildungen $F_{sn_1\ldots n_r}$ als eine wohlgeordnete Spezies $\varphi_{sn_1\ldots n_r}$ erzeugt wird, von welcher die Urspezies einkehrenden (d.h. versicherten, aber keinen versicherten echten Abschnitt besitzenden) Elementen von τ und die sonstigen konstruktiven Unterspezies Elementen von σ entsprechen. Umgekehrt entspricht in dieser Weise ein Element von σ oder ein einkehrendes Element von τ dann und nur dann einer konstruktiven Unterspezies oder Urspezies von $\varphi_{sn_1\ldots n_r}$, wenn es eine Indexreihe $sn_1 \ldots n_r p_1 \ldots p_u$ besitzt, und zwar hat in diesem Falle die betreffende konstruktive Unterspezies oder Urspezies als solche die Indexreihe $p_1 \ldots p_u$.

Das Ergebnis dieses § lässt sich wie folgt zusammenfassen:

Theorem 1. *Wenn jedem Elemente einer Menge M eine natürliche Zahl β zugeordnet ist, so ist M durch diese Zuordnung in eine wohlgeordnete Spezies S von Teilmengen M_α zerlegt, deren jede durch ein endliches Anfangssegment von Wahlen bestimmt ist. Jedem Elemente desselben M_α ist dieselbe natürliche Zahl β_α zugeordnet. Die Spezies S kann mittels erzeugender Operationen zweiter Art ω konstruiert werden, deren jede der Fortsetzung eines bestimmten für M ungehemmten endlichen Anfangssegmentes von Wahlen mit einer freien neuen Wahl entspricht. Mit einer für M gehemmten neuen Wahl korrespondiert dabei für die entsprechende Operation ω eine elementlose Urspezies.*

§ 2.

Im Falle, dass M eine finite Menge ist, ist die wohlgeordnete Spezies $\varphi_{sn_1\ldots n_r}$ einer wohlgeordneten Spezies $\psi_{sn_1\ldots n_r}$ ähnlich, welche ohne Benutzung von elementlosen Urspezies und zwar in solcher Weise der oben erwähnten Konstruktion von $\varphi_{sn_1\ldots n_r}$ parallel konstruiert werden kann, dass jeder für die Konstruktion von $\varphi_{sn_1\ldots n_r}$ angewandten Operation ω bei der Konstruktion von $\psi_{sn_1\ldots n_r}$ eine endliche Anzahl von erzeugenden Operationen erster Art χ entspricht. Die wohlgeordnete Spezies $\psi_{sn_1\ldots n_r}$ ist also unter ausschliesslicher Anwendung von erzeugenden Operationen erster Art konstruierbar. Hieraus folgt aber, dass sowohl $\psi_{sn_1\ldots n_r}$ wie $\varphi_{sn_1\ldots n_r}$ *endlich* sind, und dass insbesondere für jede natürliche Zahl s die wohlgeordnete Spezies φ_s endlich ist. Mithin kann eine solche natürliche Zahl z bestimmt werden, dass ein beliebiges einkehrendes Element von μ höchstens z Indizes besitzt, so dass die einem beliebigen Elemente e

von M zugeordnete natürliche Zahl β_e durch die ersten z erzeugenden Wahlen von e vollständig bestimmt ist.

Somit haben wir bewiesen:

Theorem 2. *Wenn jedem Elemente e einer finiten Menge M eine natürliche Zahl β_e zugeordnet ist, so kann eine solche natürliche Zahl z bestimmt werden, dass β_e durch die ersten z von den e erzeugenden Wahlen vollständig bestimmt ist.*¹)

§ 3.

Wir bestimmen nun auf der X-Achse für jede natürliche Zahl ν die k_ν-Intervalle $k'_\nu, k''_\nu, \ldots k_\nu^{(s_\nu)}$, d. h. die von links nach rechts geordneten vom Einheitsintervall der X-Achse ganz oder teilweise überdeckten $\lambda_{4\nu+2}$-Intervalle. Alsdann stellt die finite Menge I der Intervallschachtelungen $k_1^{(\mu_1)}, k_2^{(\mu_2)}, k_3^{(\mu_3)}, \ldots$ (wo jedes folgende Intervall im vorhergehenden im engern Sinne enthalten ist) eine mit dem geschlossenen Einheitsintervall der X-Achse, d. h. mit der Spezies der zum geschlossenen Einheitsintervall der X-Achse gehörigen Punkte zusammenfallende Punktmenge dar.

Bei einer *vollen* (d.h. für jeden Punkt des geschlossenen Einheitsintervalls definierten) *Funktion* $y = f(x)$ ist jeder Intervallschachtelung $k_1^{(\mu_1)}, k_2^{(\mu_2)}, \ldots$ eine Schachtelung von λ-Intervallen der Y-Achse

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¹) Dieses Theorem 2 ist implizite in der in Bd. XIII Nr. 2 der Verhandelingen dieser Akademie (1. Sektion) auf S. 4 befindlichen Formel (II) enthalten. Zur der dortigen summarischen Erörterung zu Grunde liegenden Formel (I) ist zu bemerken, dass der durch diese Formel (I) gegebene Ausdruck für $a_\varepsilon(s)$ sich weiter kompliziert, wenn die Zahl k keine Konstante ist, sondern ihrerseits wiederum durch einen Ausdruck der Gestalt des rechten Gliedes von (I) gegeben wird, und dass diese Komplikation sich dem Inhalte des obigen § 1 entsprechend in mannigfacher Weise ausdehnen lässt, ohne dass dadurch die endliche Darstellbarkeit von $a_\varepsilon(s)$ beeinträchtigt wird. Die Form von (II) wird von derartigen weiteren Komplikationen von (I) nicht beeinflusst.

Ich benutze diese Gelegenheit, um darauf hinzuweisen, dass in Z. 4 der auf der hier zitierten Seite befindlichen Fussnote statt von *Ausfüllungselementen*, von *Lokalisierungselementen* gesprochen werden muss. Dabei ist ein Lokalisierungselement r von μ'' eine jedenfalls ein Element besitzende Spezies von in unbegrenzter Fortsetzung begriffenen Folgen $F_\alpha, F_{\alpha+1}, F_{\alpha+2}, \ldots$ (α eine für r bestimmte natürliche Zahl), wo jedes F_ν ein i''_ν und jedes $F_{\nu+1}$ in F_ν enthalten ist, während F_ν für jedes ν zu einer Species S_ν gehört, von der je zwei Elemente ein Element von s''_ν gemeinsam haben.

Weiter sind auf S. 6 der zitierten Abhandlung in Z. 10 nach dem Worte "dass" die Worte: "für eine bestimmte gegen Null konvergierende Fundamentalreihe von Breiten der $\varkappa^{(\sigma)}$" einzufügen.

$\lambda', \lambda'', \lambda''', \ldots$ zugeordnet, und zwar besteht auf Grund des Theorems 2 für jede natürliche Zahl ν eine solche natürliche Zahl m_ν (von der wir voraussetzen dürfen, dass sie mit wachsendem ν nicht abnimmt), dass $\lambda^{(\nu)}$ durch die Wahl von $k_1^{(\mu_1)}, k_2^{(\mu_2)}, \ldots k_{m_\nu}^{(\mu_{m_\nu})}$ bestimmt ist. Für jedes ν kann mithin nur eine endliche Anzahl l_ν von λ-Intervallen der Y-Achse als $\lambda^{(\nu)}$ auftreten, und es besteht für dieselben eine maximale Breite b_ν, die für unbeschränkt wachsendes ν gegen Null konvergiert.

Bezeichnen wir mit $t_\nu^{(\rho)}$ das zu $k_\nu^{(\rho)}$ konzentrisch liegende Intervall der X-Achse, dessen Breite $\frac{3}{4}$ der Breite von $k_\nu^{(\rho)}$ beträgt, und seien P_1 und P_2 zwei beliebige Punkte des geschlossenen Einheitsintervalls der X-Achse, deren Abstand $< 2^{-4\nu-3}$, d.h. $< \frac{1}{4}$ der Breite der k_ν-Intervalle ist. Alsdann kann ein $t_\nu^{(\mu_\nu)}$ bestimmt werden, in dem P_1 und P_2 beide enthalten sind, und mittels dieses $t_\nu^{(\mu_\nu)}$ eine mit P_1 zusammenfallende Intervallschachtelung $k_1^{(\mu_1)}, \ldots, k_\nu^{(\mu_\nu)}, k_{\nu+1}^{(\sigma_1)}, k_{\nu+2}^{(\sigma_2)}, \ldots$ und eine mit P_2 zusammenfallende Intervallschachtelung $k_1^{(\mu_1)}, \ldots, k_\nu^{(\mu_\nu)}, k_{\nu+1}^{(\tau_1)}, k_{\nu+2}^{(\tau_2)}, \ldots$

Sei ε eine beliebige positive Grösse. Wählen wir ν_ε so gross, dass $b_{\nu_\varepsilon} < \varepsilon$ und setzen wir $2^{-4m_{\nu_\varepsilon}-3} = a_\varepsilon$, so gehören nach dem zweiten Absatze dieses § zu zwei beliebigen Elementen von I, für welche $\mu_1, \mu_2, \ldots \mu_{m_{\nu_\varepsilon}}$ gleich sind, zwei Werte von f, deren Differenz weniger als b_{ν_ε}, mithin weniger als ε beträgt. Nach dem dritten Absatze dieses § gehören also auch zu zwei beliebigen Punkten P_1 und P_2 des geschlossenen Einheitsintervalls der X-Achse, deren Abstand $< a_\varepsilon$ ist, zwei Werte von f, deren Differenz weniger als ε beträgt, so dass f sich als *gleichmässig stetig* herausstellt und wir bewiesen haben:

Theorem 3. *Jede volle Funktion ist gleichmässig stetig.*[2])

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[2]) Theorem 3 sowie eine Skizze seines Beweises befinden sich schon auf S. 5 der in Fussnote [1]) zitierten Abhandlung.

Zoology. —"*On the development of the larva of the fourth stage of Strongylus vulgaris (Looss)*". By Prof. J. E. W. IHLE and Dr. G. J. VAN OORDT. (Communicated by Prof. C. PH. SLUITER).

(Communicated at the meeting of March 29, 1924).

The investigations of MAUPAS, LOOSS, SEURAT and others showed that many Nematodes undergo 4 moults during their larval development; consequently 5 stages can be distinguished in the post-embryonic life. During the 5th (imaginal) stage no changes occur in the body-structure; the worm gradually becomes adult.

Though the development of *Strongylus vulgaris* (Looss), syn. *Sclerostomum bidentatum* Sticker, a common parasite inhabiting the large intestine of the horse, has been investigated repeatedly, the number of moults occurring in this species, is not precisely known. The investigations of DE BLIECK (1923) and BAUDET showed that the free-living larva undergoes two moults, so that the horse infects it self *per os* with larvae of the third stage. As is often the case in Nematodes, this larva is enveloped by the cuticle, which was not shed during the second ecdysis (encysted larva). Moreover DE BLIECK and BAUDET stated that this larva, penetrating the wall of the intestine, sheds its cuticle. It is not known what happens to the larvae after the horse is infected. It is supposed that they are transmitted by the blood to their definitive seat in the arteries, especially in the Arteria mesenterica cranialis, where they cause the aneurysmata, so often mentioned in literature. This supposition is supported by the fact that several investigators found in liver and lungs small knobs of the size of the head of a pin, in which Nematode-larvae, some of which were calcinated, were found. These are considered to be larvae of *Strongylus vulgaris* (OLT, 1895, pp. 363, 364, ADELMANN, 1908, p. 16), which stay in liver or lungs and die there. However, we could not find any description about their structure in literature.

It is certain that the larvae living in the aneurysmata undergo one more moult, which takes place during their presence in the aneurysmata. For this reason we may consider these larvae to belong to the 4^{th} stage. So it is not known how the larva of the 3^d stage, which infects the horse, passes over into the larva of the 4th stage, which lives in the aneurysma and where the 3d moult takes place.

In the following pages some particulars about the structure of the larva of the 4th stage and of the 4th ecdysis will be mentioned; we especially paid attention to the imaginal mouth-capsule, which develops in this larva, but which does not function till after the last moult.

The numerous larvae investigated, found in aneurysmata, were collected by the commission, appointed to inquire into the Sclerostomiasis in Holland. They were fixed in hot alcohol and rendered transparent in creosote afterwards.

Though larvae of the 4th stage are repeatedly mentioned in veterinary literature, the structure of them has not been completely investigated from a zoological point of view; cf. LEUCKART (1876, pp. 447—454) and STICKER (1901*a*). LEUCKART especially traces the development of the mouth-capsule. STICKER investigated the development rather circumstantially. Besides the free-living stages he distinguishes a young and an old larval stage (p. 211). According to STICKER the "young larva" with mouth-rosette passes over into the old larva by means of a moult and the old larva into the adult worm by means of another ecdysis. However, we never observed the ecdysis last-mentioned, nor did we find any proof of it in the publication of STICKER. Consequently the "older larva" of STICKER must be the young worm, not yet sexually mature.

The smallest (male) larva of the 4th stage found by us in an aneurysma measures 3.8 mm. in length; so it is somewhat smaller than the smallest larvae (4 mm. in length), mentioned by STICKER (1901*b*, p. 54). Female larvae are larger than male ones of the same age, the former measuring 16—17 mm., the latter 11.5—13 mm. in length, just before moulting.

This larva is characterized by the possession of a round mouth-opening, surrounded by a six-lobed rosette, of which the margin is delicately denticulated (cf. STICKER, 1901*a*, p. 193; MARTIN, 1900, p. 22 (thesis), fig. III). This rosette is connected with the cuticle of the mouth-margin, but peripherally it is separated from it by a thin layer, as is visible in the optic section. The short funnel-shaped mouth-cavity conducts directly to the oesophagus. We did not observe an independent mouth-vesicle, mentioned by STICKER (1901*a*, p. 193), nor did we see a provisional mouth-cup, as LEUCKART (1876, pp. 450, 451) did. The oesophagus is cylindrical in shape and measures about 1.2 mm. in length in the largest specimens. The anterior pointed extremity of the oesophagus is surrounded by a cuticular collar, originating from the wall of the mouth-cavity. The posterior margin of this oesophagus-collar is thicker than the anterior one. This collar is called by Sticker the "tiefere Schicht" of the "transparente Gebilde" (1901*a*, p. 193, in his figures indicated by *c*).

Male and female larvae differ in the structure of the posterior end. In the ♂ the distance from the anus to the posterior extremity is much shorter than in the ♀. Hence the posterior end of the ♂ is more obtuse than that of the ♀ (fig. 1).

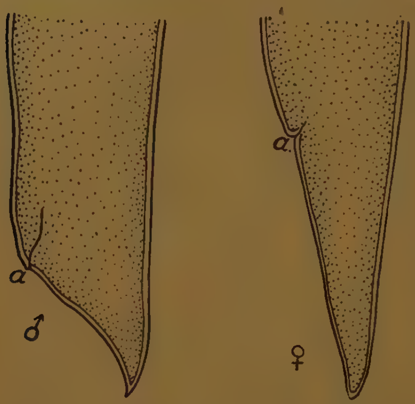

Fig. 1. Posterior end of a young male larva (long 5.6 mm.) and of a young female larva (long 5.1 mm.) of *Strongylus vulgaris* (× 183).

The first changes, connected with the formation of the imaginal mouth-capsule, are found already in very young larvae. Peripherally of the oesophagus-collar, close behind the mouth-rosette, a circular slit-shaped cavity arises (fig. 2). Between the oesophagus-collar and this cavity is situated a fine granular substance, called by Sticker "oberflächliche Schicht" of the "transparente Gebilde" (Sticker 1901*a*, figs. 13, 14, 15, (c_1)). Gradually this cavity increases in size by extension in caudal direction. Then a second cavity develops, directly behind the anterior one, separated from it by a thin membrane (fig. 3). This septum is situated at the level of the posterior margin of the

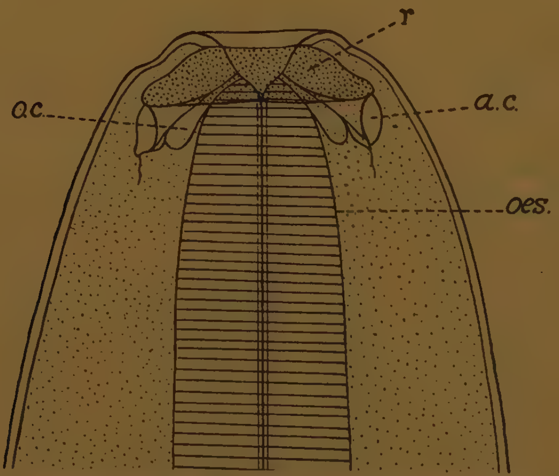

Fig. 2. Anterior end of a female larva of *Strongylus vulgaris*, long 9.1 mm. (× 275).

oesophagus-collar. This posterior cavity is the rudiment of the definitive mouthcapsule-cavity, while the mouth-collar will develop

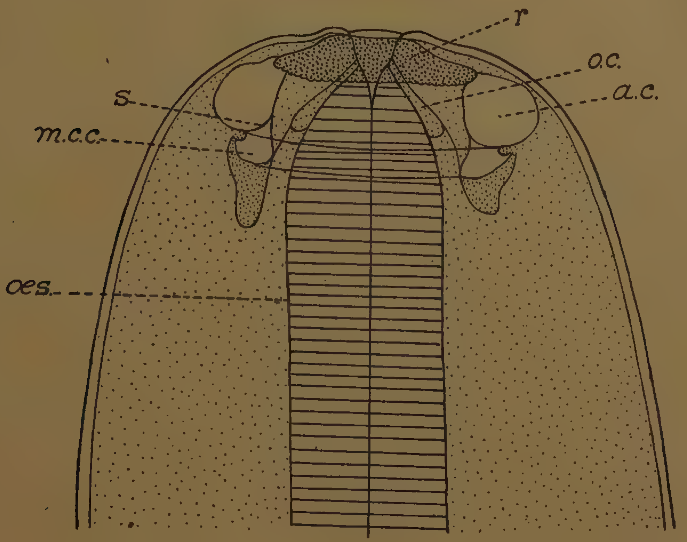

Fig. 3. Anterior end of a female larva of *Strongylus vulgaris*, long 16 mm. (× 210). The rudiment of the definitive mouthcapsule-cavity is visible.

peripherally at the level of the septum. At the posterior side of this second cavity a granular substance, possibly the material from which the imaginal mouthcapsule-wall will develop, is found. In older larvae (fig. 4) the rudiment of the mouthcapsule-wall and of the

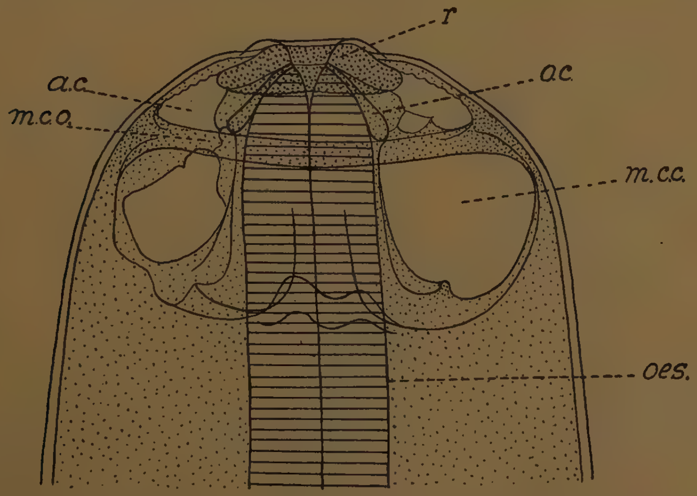

Fig. 4. Anterior end of an older female larva of *Strongylus vulgaris*, long 15.5 mm. (× 170). The definitive mouth-capsule cavity and the mouth-collar are clearly perceptible.

mouth-collar are clearly perceptible. The septum, above mentioned, situated at the level of the mouth-collar, still separates the lumen of the definitive mouth-capsule from the anterior cavity.

We direct the attention to the agreement in the development of the mouth-capsule in *Strongylus* and in *Cylicostomum*: in the latter we also found (1923) this anterior cavity situated in front of the adult mouth-capsule.

From the above it follows that we cannot agree with the opinion of STICKER on the development of the mouth-capsule. This investigator (1901*a*, p. 194) derives the exterior leafcrown (STICKER'S "Wimperkranz") from the "oberflächliche Schicht" of the "transparente Gebilde" (the fine granular substance around the oesophagus-collar) and the rudiment of the mouth-capsule from the "tiefere Schicht" of the "transparente Gebilde" (oesophagus-collar).

As we said before, LEUCKART also traces the development of the mouth-capsule. He communicates (1876, p. 451) that a circular slit-shaped cavity occurs of which the posterior part becomes the lumen of the mouth-capsule and of which the anterior part corresponds to our "anterior cavity". A septum is not mentioned by LEUCKART.

Before the ecdysis itself takes place, the definitive mouth-capsule

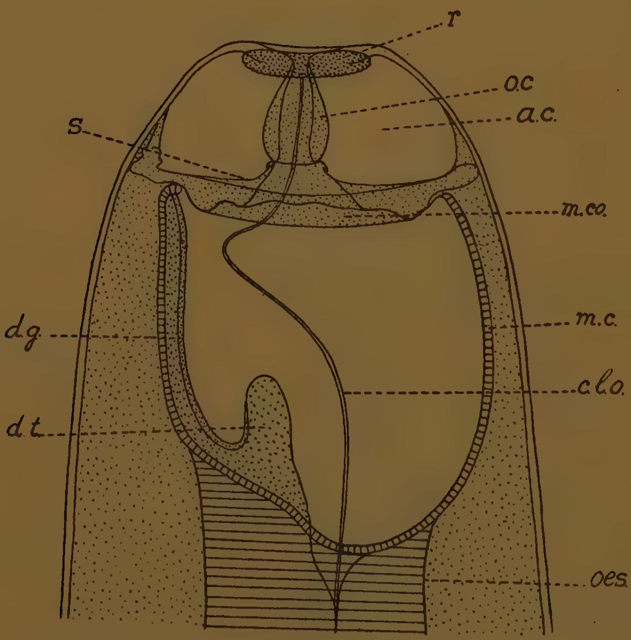

Fig. 5. Anterior end of an old female larva of *Strongylus vulgaris*, long 17 mm. (× 123). The definitive mouth-capsule is present; the larval mouth-rosette and the oesophagus-collar have separated from the oesophagus.

is already completely formed (fig. 5). Then the dorsal gutter and the double tooth at the posterior end of the gutter are also present. Still a septum divides the anterior cavity from the cavity of the mouth-capsule. However, the anterior extremity of the oesophagus has loosened itself from the mouth-rosette and the oesophagus-collar: it has moved backward and forms the continuation of the mouth-capsule. Now this collar encloses a small cylindrical space; at the anterior side it is still connected with the mouth-rosette and at the posterior side with the septum. The internal cuticular lining of the oesophagus of the larva does not participate in the backward movement above mentioned and connects the mouth-rosette with the anterior extremity of the oesophagus.

Moulting specimens have been described several times. The young animal does not completely fill up the shed cuticle, which projects a long way at both ends of the worm (cf. LEUCKART, 1876, fig. 258; FIEBIGER, 1912, fig. 200). At this moment the cuticular lining of the oesophagus of the larva is still connected with the old cuticle, which is still provided with the mouth-rosette and the oesophagus-collar, now swollen.

Finally we remark that contrary to M. SCHLEGEL (1907, p. 52), OLT (1900, p. 393) and F. ADELMANN (1908, p. 26) we agree with the current opinion, which holds that the passage of the larvae of *Strongylus vulgaris* necessarily goes through the bloodstream. Hence the larvae, which are found in the aneurysma are by no means stray specimens. This is evident from the observation that larvae provided with a rosette, are exclusively found in the aneurysmata and not anywhere else in the body of the horse. If the passage through the bloodstream were not necessary, the larvae of the 4th stage would be met with in other places as well and this is never the case, a fact we can establish after having investigated an extensive material.

Utrecht, March 1924. *Zoological Laboratory, School of Veterinary Medicine.*

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ABBREVIATIONS.

a. = anus.
a.c. = anterior cavity.
c.l.o. = cuticulair lining of the oesophagus.
d.g. = dorsal gutter.
d.t. = dorsal tooth.
m.c. = mouth-capsule.
m.co. = mouth-collar.
o.c. = oesophagus-collar.
oes. = oesophagus.
r. = mouth-rosette.
s. = septum.

Physiology. — *"A Contribution concerning the Function of the Vestibular Apparatus"*. By Prof. R. MAGNUS and A. DE KLEYN.

(Communicated at the meeting of December 29, 1923).

We have demonstrated in a previous publication [1]) that for more than a week after the removal of the otolithic membranes the sensory epithelium of the maculae is still in a constant condition of stimulation. When in guinea-pigs, deprived of all otolithic membranes after WITTMAACK's method, one labyrinth is for some time eliminated by cocain, the stimuli emanating from the non-injected labyrinth will induce asymmetrical phenomena, similar to those after unilateral extirpation of the labyrinth in normal animals. There is however one difference between animals so treated and normal animals: in the centrifuged animals injected unilaterally with cocain these phenomena do not change with a change of position of the head in space.

Considering that there was a week's wait after the centrifugation, it is highly probable that the above condition of stimulation should no longer be ascribed to centrifugation. It was concluded therefore, that to the sensory epithelium of the maculae the power should be assigned of emitting stimuli, the strength of which does not vary in consequence of the absence of otolithic membranes. The function of otolithic membranes, then, consists in altering the intensity of this condition of stimulation of the sensory epithelium. The stimulation will be stronger or weaker according as the membranes pull at the epithelium or exert a pressure upon it.

Now it struck us in these and also in other experiments, in which cocain was injected unilaterally into the middle-ear of normal guinea-pigs, that, whenever the phenomena of paralysis of the labyrinth manifested themselves, first a vertical eye-deviation was noticed without a trace of nystagmus, combined with a rotation of the head towards the injected side ("Grunddrehung"), [2]) and that only much later, sometimes only after 20 minutes, a nystagmus appeared with the quick component phase towards the non-injected side. Since the vertical eye-deviation and the "Grunddrehung" are to be considered as due to a paralysis of the sacculus, resp. utriculus [3]) we accordingly attributed the nystagmus, which appeared so much later, to a paralysis of the semicircular canals. That this hypothesis is correct can be readily ascertained experimentally by showing that in the first stage in which eye deviations and "Grunddrehung" of the head occur, the semicircular canals on the injected side (through rotation and calorically) are still responsive to stimuli, and that they lose this property when the nystagmus appears. Lower down we shall see, that this is a fact. It also became evident that the sacculus and the utriculus maculae are not paralyzed simultaneously, but that there is a stage, in which the utriculus still maintains its normal activity, while the sacculus is already paralyzed. The following results were obtained in nineteen successful experiments with guinea-pigs nine of which were normal, whilst in ten the labyrinths had been extirpated — in five of these only a few hours and in five 5 days previous to the experiment. Before the experiment was started, the various labyrinth-reflexes (tonic as well as those of the semi-circular canals) were carefully examined in all the animals. Only when these proved quite normal was the animal selected for the experiment.

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[1]) R. MAGNUS and A. DE KLEYN. "A further Contribution concerning the function of the Otolithic apparatus". These Proceedings Vol. XXV, p. 256.

[2]) "fundamental rotation".

[3]) R. MAGNUS and A. DE KLEIJN. "The Function of the Otolithes". These Proc. Vol. XXIII, p. 907.

I. *Experiments with normal animals.*

On injecting the (e.g. left) middle ear of a normal guinea-pig through the meatus acusticus externus with 0.1 c.c. of a 5°/$_0$ cocain-solution, we shall find that after a few minutes (on an average 4 to 6) first the labyrinth righting-reflex of the head becomes weaker and soon disappears altogether when the animal is held in left lateral position in the air; then the animal holds its head in left-lateral position, whereas, when the animal is examined in right-lateral position in the air, the head directly assumes its normal position. At the same time or directly afterwards a vertical eye deviation (without nystagmus) appears; the left eye is turned downwards (towards the belly), the right eye upwards (towards the back). A maximal deviation occurs with the right lateral position of the head. It diminishes with the normal position of the head, and is minimal or zero with the left lateral position, so that it is entirely similar to the vertical eye deviation subsequent to levolateral labyrinth extirpation.

The absence of the labyrinth righting reflex, when the animal is in left lateral position, and the vertical eye deviation above-described are symptoms of a paralysis of the left sacculus macula. In this stage there is not yet a trace of "Grunddrehung", which becomes quite evident when the animal is examined with the head downwards.

The animal sits normally and can walk forwards steadily without deviation, which implies that the injection has not yet affected the utriculus.

It is interesting to note that the Labyrinth righting-reflexes, which enable the animal to correct abnormal, but symmetrical positions, and thus to bring the head into the normal position (lower jaw down, mouth-fissure about 30° under the horizontal), are still normally active in this stage. When the animal is examined with the head upward, the head will be brought into its normal position by flexion in the ventral direction. If the animal is held with the head downwards, the head is flexed in the dorsal direction. If the animal is held in back position, the animal achieves the same result by bending the body forwards.

These facts lend support to the conception that these "symmetrical" labyrinth righting-reflexes are provoked in the utriculi, as has already been assumed in a previous publication.

In this stage all the reactions of the head and eye to rotation, and their accompanying nystagmus were present normally on each side and in each direction.

After varying intervals (sometimes after 7—14 minutes, sometimes sooner) it will be seen that in an examination with the head downwards a "Grunddrehung" appears towards the injected (left) side. At first it is insignificant, but soon it becomes more intense and at length it amounts to 90° and more. It is clear now that an utriculus paralysis has appeared. When the animal sits normally, the head is also turned to the left and the body inclines towards the left.

On the other hand there is total absence of flexion of the head towards the left, so that the animal walks straight and only occasionally tumbles to the left, if the "Grunddrehung" is intense.

In this stage a spontaneous nystagmus of the head or eye is not noticeable and we could make the following experiments to demonstrate that the semicircular canals on the injected side were still performing their proper function.

The rotation-reactions of the head are present with rotation to the right as well as to the left, and *completely symmetrical.* The same can be said regarding the horizontal, vertical and rotatory eye-deviations with their accompanying nystagmus (as the result of similar tests on the corresponding canals).

The disadvantage of these experiments, however, is, that during the rotation both labyrinths are always stimulated; they will be followed, however, by supplementary experiments in which, after unilateral labyrinth-extirpation, the normal labyrinth was injected with cocain, so that the observed labyrinth-reflexes could be elicited only in this labyrinth.

The only appropriate method to carry out a purely isolated examination of the semicircular canals of one labyrinth alone, is douching the meatus with cold or warm water (caloric examination after Bárány). If the nystagmus, which then follows, has the *opposite* direction with the head down to that with the head in normal position, it is certain that it has been elicited by the caloric stimulus. These caloric experiments yield conclusive evidence only when the nystagmus disappears again after the douching has been discontinued, and does not pass into the continuous, spontaneous nystagmus by which the next stage is characterised. In the experiments alluded to above this appeared not to be the case.

The third stage, i.e. the paralysis of the semicircular canals develops sometimes very quickly, in other cases only after 7—8 minutes. The head is then flexed to the left, at first slightly, but gradually more distinctly, and a flexion of the body to the left accompanies it. The rotation-reaction of the head is getting asymmetrical; while it is unaltered in the direction of the flexion of the head, it gets weaker and weaker in the opposite direction. The spontaneous nystagmus, which consisted at first in a few beats with varying direction, presently becomes regular, forcible and the left eye beats quickly upwards and towards the nose, while the right eye beats downwards and towards the ear. That it is typified as a "paralyzing nystagmus" is evidenced by the fact that there is no change of direction whether the animal is examined with its head in normal position or in the "head-down" position. By rotation to the right, or to the left, this spontaneous eye-nystagmus is influenced asymmetrically in exactly the same way as the spontaneous nystagmus after unilateral labyrinth-extirpation. Typical spontaneous head-nystagmus can also be noted now, and in consequence of the flexion of head and trunk the animal can no longer walk straight forwards but moves clockwise.

Thus in these experiments the complete elimination of the labyrinth takes place in three stages.

The first stage, in which the paralysis of the sacculus macula occurs, is characterized by the disappearance of the labyrinth righting-reflexes with one of the lateral positions of the animal in the air and the occurrence of a vertical eye-deviation; in the second stage the utriculus paralysis appears in a "Grunddrehung" towards the injected side but without lateral flexion, while in the third stage also the semicircular canals are paralysed and the picture of the total elimination of one labyrinth is completed by the lateral flexion of the head with spontaneous head- and eye-nystagmus.

Only one symptom of the unilateral labyrinth extirpation was missing in all these experiments, viz. the decrease of tonus of the extremities on the injected side. Even after complete paralysis of the labyrinth by the cocain, which in one case was accompanied by paresis also of the facialis, we never observed that the left extremities were more flaccid than the right ones, when the head was brought in a symmetrical position relative to the trunk. As regards this symptom, which can nearly always be observed after extirpation of the labyrinth, we are still left in the dark.

Of one of the experiments under consideration the following typical account may be recorded here:

Guinea-pig IV, spotted.

Prior to the experiment all the labyrinth reflexes are normally present, and symmetrical, (*rotation-reactions of the head and of the eye*, horizontal, vertical and rotatory, with the accompanying nystagmi; *reactions to progressive movements:* lift-reaction, springing-reflex and extension of the toes; *all labyrinth righting reflexes* from the lateral as well as from symmetrical positions; *compensatory eye-positions*: vertical and rotatory).

The animal sits symmetrically, eyes symmetrical in normal position without nystagmus. Tonus of the fore-limbs equal. Hearing-reaction normal.

11^{9}. 0,1 cc. 5 $^{0}/_{0}$ cocain injected into left middle-ear.

11^{12}. Labyrinth righting-reflexes on either side positive and symmetrical.

11^{13}. Beginning of eye deviation: the left eye deviates downwards.
Labyrinth righting-reflex in the right lateral position positive (the head is brought into normal position), but in the left lateral position negative (head remains in left lateral position). Not a trace of "Grunddrehung".

11^{14}. Vertical eye deviation: left eye downwards, right eye upwards.
Tonus of fore-limbs equal or slightly weaker in the right fore-limb.
Rotation-reaction of the eye and nystagmus (horizontal as well as vertical and rotatory) with dextro-, and levo-rotation: on either side positive and symmetrical. No "Grunddrehung", no spontaneous nystagmus.

11^{17}. Now distinct "Grunddrehung" to the left.

11^{19}. Douching of left meatus with some cold water: distinct nystagmus: the left eye beating towards the nasal canthus and upwards. With head-down position: the left eye beating towards the aural canthus and downwards.

11^{30}. Caloric nystagmus gone, no spontaneous nystagmus.
Rotation-reactions of the eye and nystagmus ((horizontal, vertical and rotatory): distinct and symmetrical.
Rotation-reactions of the head with dextro-, and levo-rotation also symmetrical in spite of head-rotation ("Grunddrehung").
Labyrinth righting-reflexes in right lateral position positive, in left lateral position absolutely negative.
Tonus fore-limbs in dorsal position (with the positions of the head and trunk in their normal relationship): right fore-limb more flaccid.

11^{25}. Intensive vertical eye deviation, no spontaneous nystagmus.

Tonus fore-limbs: right fore-leg more flexed; resistance to flexion on the left or the right leg no marked difference.

11^{27}. Left eye deviated downwards and slightly forwards; right eye upwards, but no deviation towards the ear. "Grunddrehung" 90°.

11^{32}. First appearance of a spontaneous nystagmus: with normal position of thc head in the left eye upwards and towards the nose, in the right eye downwards and towards the ear. In examination with "head-down" no change of the nystagmus, only a slight alteration of the direction corresponding with the compensatory eye-position in this case.

11^{35}. Horizontal rotation induces nystagmus of the eyes asymmetrically: during dextrorotation spontaneous nystagmus rapid, during levorotation slower; attempts to estimate the vertical and rotatory nystagmus as the result of corresponding rotation is obviated by the pre-existing spontaneous nystagmus.

Tonus fore-limbs: equal or right foreleg slightly more flaccid.

11^{40}. In addition to marked "Grunddrehung" of the head, now also marked lateral flexion of the head to the left.

Distinct asymmetry in the horizontal rotation-reactions of the head: during dextro-rotation distinct reaction, during levo-rotation no reaction visible.

Tonus fore-limbs: right fore-leg more flaccid; but resistance to flexion on the right and the left side not distinctly different.

In the experiments of this first series the following particulars were also noted: It may occur that the cocain paralysis is preceded by a stimulation of only a few minutes. In these cases the eye-deviation (sacculus) is reversed, or "Grunddrehung" (utriculus) is towards the opposite side.

In one of the experiments paralysis only of the sacculus revealed itself, while the utriculus and the semicircular canals were not affected- In a few experiments injection of cocain into the middle ear did not induce paralysis of the labyrinth at all. In such cases the dissection always revealed otitis media purulenta with a marked swelling of the mucous membrane of the middle-ear, which prevented the cocain from penetrating.

The interval between the several stadia varies considerably: in one experiment it was so short that a detailed examination of the various labyrinth-reflexes was not possible. In other cases again it took half an hour before the paralysis of the labyrinth was complete, so that a careful investigation could be made.

Once the "Grunddrehung" began very soon (4 minutes) after the injection, so that the second stage could not be distinguished from the first, as could be done in all other cases.

As stated above, the spontaneous nystagmus after unilateral elimination of the semicircular canals does not change with different positions of the head in space. When, however, the head is brought from the normal position into the "head-down" position, a slight change in the direction of the nystagmus may be observed. This is due to a change in the position of the eyes (upper corneapole moving posteriorly) consequent on compensatory rotations starting from the normal labyrinth. But this insignificant change of direction can never be mistaken for the real change in the direction of the nystagmus, as occurs in alteration of the head position during the caloric test.

In all the above experiments the labyrinth-reflexes were quite normal again the next day, when no effect whatever could be observed of the cocain injection.

From the results of this first series of experiments several conclusions may already be deduced, of which we mention the following:

1. The paralysis of the labyrinth takes place in three stages:

The first stage comprises the "asymmetrical" labyrinth righting-reflexes [1]), and the vertical compensatory eye-positions, which confirms the opinion stated before, that both are provoked in the sacculus macula. To the second stage belong the "Grunddrehung" and the "symmetrical" [2]) labyrinth righting-reflexes. As stated before the "Grunddrehung" is due to the elimination of one utriculusmacula; it is, therefore, probable that the latter plays a prominent part in the "symmetrical" labyrinth righting-reflexes.

The third group: nystagmus of head and eyes as well as the lateral flexion of the head are elicited in guinea-pigs by the semicircular canals. Now, whereas the lateral flexion of the head is a lasting symptom in rabbits and apes after unilateral labyrinth-extirpation and it is possible, therefore, that in these animals also the otolithic apparatus comes into play in the genesis of this phenomenon, this flexion is only transitory in guinea-pigs, dogs and cats after unilateral labyrinth-extirpation. In connection with the above-described experiments with guinea-pigs it is evident, therefore, that the phenomenon has to be considered as a symptom of the semicircular canals.

It also appears that elimination of the otolithic maculae (without stimulation) does not result in nystagmus and that the spontaneous head- and eye-nystagmus are to be looked upon as a symptom of the semicircular canals.

2. When the spontaneous nystagmus is not dependent on the otolithic maculae and appears when without stimulation the apparatus of the semicircular canals on the one side is eliminated, it follows that the cause of this spontaneous nystagmus is to be looked for in stimuli that owe their origin to the semicircular canals of the unimpaired side. The nystagmus is always present (in the first few days after unilateral labyrinth-extirpation), even when the head is motionless. Consequently the conclusion forces itself upon us, that there is a constant flow of stimuli from the sensory epithelium of the cristae and that the cupulae, when they are moved or pressed by the endolymph, reinforce or weaken these constant stimuli elicited in the cristae. This leads us to the same conclusion we previously drew for the otolithic maculae, viz. *that it is essentially a sensory epithelium producing constant stimuli, while the overlying apparatus exerts an influence upon the intensity of those stimuli in a positive or*

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[1]) The labyrinth righting-reflexes from lateral positions of the animal.

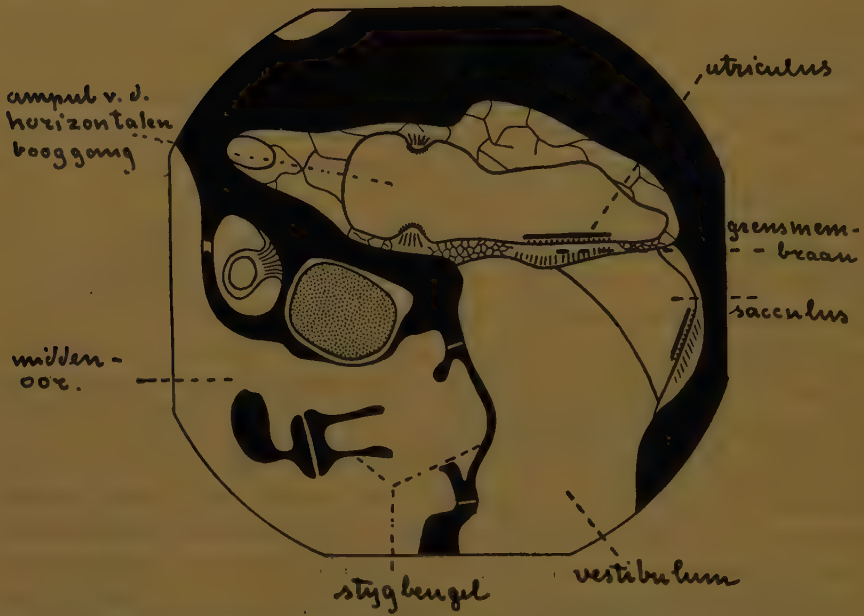
[2]) The labyrinth righting-reflexes from symmetrical positions of the animal.

in a negatieve sense. This fact may account for the extreme sensitiveness of the sensory epithelium in the labyrinth.

The mechanical relations in the semicircular canals are such that modifications of the condition of stimulation of the sensory cells appear only for some time, whereas with the otolithic maculae permanent modifications are possible. *Reflexes* of the semicircular canals are always transitory; the sense-organ, however, transmits constant stimuli to the centre. This conception is closely allied to EWALD's "Tonus-labyrinth".

3. What strikes us as most important is that in guinea-pigs the symptoms of the unilateral labyrinth-extirpation, provoked in the intact semicircular canals of the opposite sides, such as head-, and eye-nystagmus, lateral flexion of head, and trunk and clockhand movements, are all transitory, and will disappear after a few days, whereas the otolithic symptoms are permanent.

Fig. 1.



ampul v. d. horizontalen booggang = ampulla of the horizontal semicircular canal.
middenoor = middle-ear.
utriculus = utriculus.
grensmembraan = membraneous partition.
sacculus = sacculus.
vestibulum = vestibulum.
stijgbeugel = stapes.

4. The order in which the various sensory cells are paralyzed by the cocain injected into the middle-ear corresponds entirely to the anatomic relations.

Fig. 1 represents the middle-ear separated from the vestibulum by a thin wall in which the foramen ovale and stapes are situated. At the medial wall of the vestibulum the sacculus may be seen. Utriculus and semicircular canals are separated from it by the membraneous partition (de Burlet), and consequently screened from the effect of cocain. That the utriculus is affected by the cocain before the semicircular canals, may perhaps be explained by the fact that the utriculus is in immediate proximity to this membraneous partition.

The above conclusions are further corroborated and augmented by the following series of experiments.

II.

Five days before the experiment proper the right labyrinth was extirpated in five guinea-pigs. After these five days the typical complex of symptoms was found.

The head is rotated downwards and to the right; when the animal sits, the head is flexed to the right or held straight, it may also be flexed to the left. The rotation reactions of the head like the horizontal rotation-reactions of the eye with accompanying nystagmi are asymmetrical. Vertical and rotatory reactions of the eye are present. Spontaneous nystagmus is absent or present only in a slight degree. The reactions of the progressive-movements are positive. "Grunddrehung" ± 90° to the right. There is the typical picture of the labyrinth righting-reflexes: with right-lateral position of the animal, the head remains in lateral position; from left-lateral position it is brought into the normal position. The right eye deviates downwards (sometimes slightly posteriorly), the left eye upwards (sometimes slightly anteriorly). This eye-deviation is maximal in the left lateral position of the head, less considerable in the normal position and minimal in the right-lateral position. The right fore-limb is more flaccid than the left one, when the head and trunk are maintained in their normal relative positions.

When 0,1 c.c. 5 °/₀ cocain-solution is injected into the *left* middle-ear of such an animal the effects of the right-lateral labyrinth extirpation will be seen to disappear in a definite order corresponding to the gradual paralysis of the different sensory cells of the left labyrinth. Here also we observe first of all a paralysis of the *sacculus*. Accordingly with left-lateral position of the animal the head in the air is no longer held in left-lateral position but in back position ("Grunddrehung"). At the same time, or a short time afterwards, there is a decrease in the difference between the vertical eye-deviation with right-, and left-lateral position, which ultimately disappears altogether; the eyes are then, in normal position both in the right- and the left-lateral position of the head, or in both lateral positions they deviate but little in the original direction (probably under the influence of the left semicircular canal apparatus). In this stage the "Grunddrehung" is still 90°; spontaneous nystagmus is absent, the rotation-reactions and the lateral flexion of the head have not altered.

After this stage the "Grunddrehung" begins to decrease, and ultimately disappears altogether, which causes the head to be held in lateral position with both lateral positions of the animal in the air, since now the sacculi as well as the utriculi are eliminated on either side.

On the other hand the left semicircular canals are still unaffected, and accordingly the head is flexed to the right (without "Grunddrehung"); the rotation-reactions of the head are asymmetrical, while the horizontal, vertical and rotatory rotation-reactions of the eye with nystagmi do not differ in any way from those before the injection.

Lastly also the semicircular canals are paralyzed. It is noteworthy that this occurs in two stages. In the first stage the dextro-flexion of the head disappears and may even be changed into a levo-flexion in consequence of the *Bechterew*-compensation. Now the horizontal rotation-reactions of the head can no more be elicited and in most cases a spontaneous *Bechterew*-nystagmus to the right arises. In this stage, however, the vertical rotation-reactions of the eye and nystagmus are most often distinctly, sometimes very distinctly visible.

Since it is generally accepted that the horizontal rotation-reactions are provoked in the horizontal semicircular canals, these facts must be explained as follows: first the cocain paralyzes the crista of the horizontal semicircular canal, whereas the cristae of one or of both vertical semicircular canals are still active. Finally also the elimination of the other semicircular canals is completed, so that we have a picture of symptoms of an animal, in which both labyrinths have been extirpated after an interval of some days. Only the greater flaccidity of the legs on the side where the first labyrinth-extirpation took place, remains. The successive paralysis of the various parts of the apparatus of the semicircular canals is shown by the successive development of a typical *Bechterew* nystagmus; in the first stage the direction is more horizontal, in the second stage it becomes more vertical. In several cases e.g. the nystagmus of the left eye was directed in the first stage horizontally and anteriorly, that of the right eye posteriorly and downwards; in the second stage the nystagmus in the left eye was directed anteriorly and upwards, whilst the nystagmus in the right eye was directed vertically and downwards.

We now subjoin a typical account of an experiment like the one described above:

Guinea pig XVI, white.

14. XI. '23. On investigation all the labyrinth-reflexes appeared to be present, the tonic as well as those due to kinetic action, all normal and symmetrical. The animal sits symmetrically, eyes symmetrical without nystagmus, no "Grunddrehung". Auditory reaction positive.

15. XI. '23. Extirpation of the right labyrinth.

20. XI. '23. When the animal sits, the head, which is turned to the right, is held straight, or flexed to the right, but may, however, also be flexed to the left.

Reflexes of the semicircular canals: Rotation-reaction of the head negative during dextro-rotation; positive during levo-rotation.

Rotation-reaction of the eye and nystagmus; (horizontal, vertical and rotatory): the same as those after right labyrinthextirpation.

Reactions to progessive movements. Lift-reaction and Springing-reflex, on both sides positive.

Labyrinth righting-reflexes: in both lateral positions, with "head-upwards" and "head-downwards" and with dorsal-position quite typical for an animal after right labyrinth-extirpation.

Vertical compensatory eye-positions: like those after right labyrinth-extirpation.

"Grunddrehung": 90°.

Right fore-limb generally more flaccid than the left one, but sometimes no distinct difference.

10^{58}. 0,1 cc. 5 $^0/_0$ cocain into left middle ear.

11. Right lateral position of the animal in the air: back of the head turned downwards ("Grunddrehung").

The difference between the vertical eye deviation in right-, and left-lateral position diminishes.

11^{1}. Right-lateral position: back of the head turned downwards. Difference of the vertical eye deviation in both lateral positions disappeared. "Grunddrehung" 90° to the right.

11^{2}. Horizontal and vertical rotation-reactions of the eye with nystagmus: positive.

"Grunddrehung 70°. When sitting head flexed to the right.

11^{4}. "Grunddrehung" 30°. Horizontal rotation-reaction of the eye: positive. No spontaneous nystagmus.

11^{6}. "Grunddrehung" disappeared. Horizontal rotation-reaction of the eye: slightly positive, a little later negative. Vertical rotation-reactions of the eye: intensely positive. No spontaneous nystagmus.

11^{7}. Head flexed to the left: no "Grunddrehung".

$11^{7\frac{1}{2}}$. Beginning of the spontaneous nystagmus, at first only aggressively.

Right eye deviated anteriorly and slightly upwards, with nystagmus posteriorly and slightly downwards: left eye deviated posteriorly with nystagmus anteriorly.

11^{9}. No "Grunddrehung". Head flexed to the left. Vertical rotation-reactions of the eye also disappeared.

11[12]. Right eye deviated upwards with nystagmus downwards; left eye deviated downwards with nystagmus anteriorly upwards.
When sitting head flexed to the left. No "Grunddrehung". When suspended "head down" only flexion of the head towards the left.
Right fore-leg more flaccid than the left one.
Left facialis intact (corneal reflex).

In one experiment there appeared only a *Bechterew*-reversal of the "Grunddrehung", while no reversal of the nystagmus or the head-flexion took place. In this case the horizontal rotation-reactions and the flexion to the right disappeared in the first stage of the paralysis of the semicircular canals. while there were still considerable vertical rotation reactions of the eye, that did not disappear before the second stage. In another case "Grunddrehung" was reversed before there were yet symptoms of paralysis of the semicircular canals. On the other hand in another experiment there appeared no reversal of the "Grunddrehung" but of the head-flexion only, while there appeared also a *Bechterew*-nystagmus of the eyes.

This again points to the independence of the symptoms of semicircular canals and utriculus; also "Grunddrehung" and head flexion behaved in these experiments independently of each other.

The next day the action of the cocain had disappeared entirely in all the cases and the left labyrinth performed its function in a normal way.

III.

When in experiments after right labyrinth-extirpation the injection of cocain into the left middle-ear was given 4 or 5 hours later, the successive disappearance of the acute symptoms could be watched easily.

The animals sat, after right labyrinth-extirpation, with considerable dextroflexion of the head and the trunk, had head-nystagmus towards the left, intense rotation of the head to the right, typical eye deviation, (on the right downwards, on the left anteriorly upwards), which was maximal with left-lateral position, and minimal with right-lateral position of the head. Eye-nystagmus on the right side anteriorly upwards, on the left side posteriorly downwards.

With left-lateral position of the animal in the air the head was turned into the normal position, i.e. with the dorsum of the head upwards; with right-lateral position, however, the head was in right-lateral position. Distinct asymmetry of the rotation-reactions of the head and of the horizontal rotation-reactions of the eyes when the animals were rotated to the left and the right. Rolling movements towards the right.

The following protocol typifies these experiments.

Guinea-pig XII, white-black.

6. XI. '23. On investigation all labyrinth-reflexes are present and symmetrical. Animal sits symmetrically, eyes also symmetrical without nystagmus. No "Grunddrehung". Auditory reaction positive.

10^{15}. Right labyrinth-exstirpation.

3. Head rotated and flexed to the right; nystagmus to the left, "Grunddrehung" 90° to the right. Right eye deviated backwards and downwards with nystagmus anteriorly upwards; left eye with deviation anteriorly upwards and nystagmus posteriorly downwards.

On examination with normal position of the head and with "head-down" no alteration in the direction or type of the nystagmus

Typical eye deviation and labyrinth righting-reflexes as those after extirpation of the right labyrinth.

Rotation-reactions of the eye also typical for right labyrinth-extirpation. Rotation-reaction of the head with dextro-rotation absent, positive with levo-rotation.

Right fore-leg more flaccid than the left one.

4^{28}. 0.1 cc. 5% cocain into the left middle-ear.

4^{30}. Right-lateral position; head 45° toward back-position.

4^{31}. Right-lateral position. Back of the head turned downwards.

Decrease of difference in eye-deviation between right-, and left-lateral position.

"Grunddrehung" 90°.

Rotation-reaction of the head still positive with rotation to the left. Evident spontaneous nystagmus.

4^{33}. Small difference in eye-deviation with right-, and left-lateral position; with normal head-position both eyes are still slightly deviated Typical spontaneous nystagmus.

4^{36}. No difference in eye-deviation with right-, and left-lateral position; with different positions of the head in space still a slight but constant deviation of the eyes in the original direction.

"Grunddrehung" 90°.

Rotation-reaction of the head with levo-rotation still positive; spontaneous nystagmus equal with both lateral positions.

4^{38}. "Grunddrehung" 70°. Shortly afterwards 45°.

Rotation-reaction of the head with levo-rotation positive; nystagmus still present.

4^{39}. "Grunddrehung" 30°.

With lateral position on the right and the left: head in lateral position.

4^{41}. "Grunddrehung" 20°.

Rotation-reaction of the head to the left distinctly positive; to the right negative.

Spontaneous nystagmus positive: right eye anteriorly upwards; left eye posteriorly downwards.

Often sits with head straight, then again head dextro-flexed.

4^{44}. When suspended, "head up" tolerates as well right-, as left-lateral position of the head.

4^{45}. "Grunddrehung" gone.

With levo-rotation rotation-reaction of the head still distinct. Spontaneous nystagmus very distinct.

When sitting inclined to flex the head to the right.

4^{49}. Eyes still slightly deviated. Nystagmus right eye upwards; left eye downwards. When sitting head flexed 30° to the right.

4^{52}. Rotation-reaction of the head with levo-rotation absent or minimal. Spontaneous nystagmus weak: right eye posteriorly, left eye anteriorly (reversal phenomenon!)

4^{53}. Eye deviation has disappeared. Right fore-leg *much* more flaccid.

4^{58}. Nystagmus of right eye posteriorly downwards; of left eye anteriorly upwards, i. e. complete reversal as compared with former results. "Grunddrehung" 10° to the right (no reversal).
Flexion of the head mostly absent, sometimes inclined to flexion to the right.

5^{3}. Nystagmus as 4^{58}. "Grunddrehung" gone. When sitting head straight.

5^{5}. Now "Grunddrehung" 45° to the left (reversal phenomenon).
When sitting rolls to the left once, but can sit up from left-lateral position.

It will be seen that in this experiment the "asymmetrical" labyrinth righting-reflexes disappeared first, so that with right-lateral position of the animal in the air the back of the head was turned downwards. At the same time the difference in eye-deviation diminished with right-, and left-lateral position and soon afterwards disappeared altogether. In this stage of paralysis of the sacculus there were still present: the "Grunddrehung", the spontaneous nystagmus, the lateral flexion of the head, the asymmetry of the rotation-reaction of the head during dextro-, and levo-rotation, and a slight deviation of the eyes in the original direction.

Then began the paralysis of the utriculus, showing itself in a diminution and ultimately in a disappearance of the "Grunddrehung".

Finally the paralysis of the semicircular canals appeared: the rotation-reaction of the head disappeared just as the lateral flexion of the head; the nystagmus was "reversed". Only a few minutes later also "Grunddrehung" was "reversed" and the animal rolled towards the left.

The direction of the spontaneous nystagmus during the paralysis of the semicircular canal was interesting:

The nystagmus was first directed	on the right: anteriorly upwards	on the left: posteriorly downwards
Shortly before the disappearance of the rotation-reaction of the head	" " " upwards (vertical)	" " " downwards (vertical)
After the disappearance of the rotation of the head (reversal of nystagmus)	" " " posteriorly (horizontal)	" " " anteriorly (horizontal)
A few minutes later.	" " " posteriorly downwards	" " " anteriorly upwards

So at the beginning of the paralysis of the horizontal semicircular canal the nystagmus that had thus far been oblique, became purely vertical. After the paralysis of the horizontal semicircular canal had been completed, the direction of the "reversed" nystagmus was purely horizontal, while after complete paralysis of the whole labyrinth it turned into the opposite direction as before the cocain-injection.

Whether the reversal of the spontaneous nystagmus in these experiments (in which there was only an interval of from 4 to 5 hours between the cocain-injection and the labyrinth-extirpation) rests on an incipient BECHTEREW-compensation or on a condition of stimulation of the trunk of the octavus on the side operated upon, cannot easily be made out, as in either case the same result can be expected.

The change in the direction of the spontaneous nystagmus at the commencement of the paralysis of the semicircular canals, was to be observed in three out of five experiments, in the way above described; twice only did a weak "reversal" nystagmus appear, which, however, in one of the experiments began as a purely horizontal movement.

It was remarkable that in this series of experiments the very first beginning of the paralysis of utriculus and of the semicircular canals was not so distinctly separated as in other experiments (described under I and II). The cause of this has not been investigated thus far.

SUMMARY.

The experiments above described, in which cocain was injected into the middle-ear of guinea-pigs, show that when the other labyrinth is unimpaired, or has been previously removed by operation, the symptoms of the cocain paralysis develop successively according to certain laws, and that the symptoms regarding the paralysis of the otolithic reflexes are in complete harmony with the suppositions previously expressed.

It is also possible to isolate another group of symptoms, which are to be referred to a unilateral paralysis of the semicircular canals. If the labyrinth on the other side is intact, these symptoms are *elicited* by cocain; if the labyrinth of the other side has been previously removed, these symptoms *disappear* in regular succession.

Paralysis of the sacculus, with the other labyrinth intact, brings on unilateral disappearance of the "asymmetrical" labyrinth righting-reflexes and vertical eye deviation without nystagmus. The "symmetrical" labyrinth righting-reflexes remain intact and consequently it is probable that they depend on the utriculi.

If the opposite labyrinth has been previously removed, the paralysis of the sacculus causes the loss of all "asymmetrical" labyrinth righting-reflexes and the disappearance of the existing vertical eye deviation (with the exception of a small trace, which most likely depends on the semicircular canals).

Paralysis of the utriculus, with the other labyrinth intact, causes a "Grunddrehung" towards the injected side, and when the other labyrinth has been previously removed, it causes the disappearance of the existing "Grunddrehung" and in some cases a "Grunddrehung" towards the other side (BECHTEREW reversal; if the other labyrinth has been extirpated only a few hours previously, this may also be caused by a stimulation of the octavus nerve on that side).

When the other labyrinth is intact neither a spontaneous nystagmus of head and eyes, nor a lateral flexion of the head will appear so long as the semicircular canals on the injected side are not affected by the cocain and still respond to caloric and rotary stimulation.

Directly when *paralysis of the semicircular canals* reveals itself, a lateral flexion of the head and spontaneous nystagmus of head and eye will be observed, while at the same time, during dextro- and levo-rotation, the rotation-reactions become asymmetrical.

If the other labyrinth has been removed beforehand, the lateral flexion of the head, the occasional slightly vertical eye deviation and the rotation-reactions on head and eyes will disappear. At the same time a spontaneous nystagmus in the opposite direction is originated.

At the beginning of the paralysis of the semicircular canals we found, that the horizontal rotation-reactions of head and eyes disappeared and that the vertical rotation-reactions disappeared evidently much later. The direction of the eye movements at the reversal of the nystagmus also points to the fact that first the horizontal and later on one vertical semicircular canal is paralyzed.

In a previous publication the symptoms observed in centrifuged guinea-pigs after injection of cocain into one middle-ear led to the conclusion that the sensory epithelium of the maculae constantly produces stimuli that vary in strength according as the otolithic membranes pull at the epithelium or exert a pressure upon it.

The experiments recorded in this paper make it highly probable that the property of producing continual stimuli, must also be assigned to the sensory epithelium of the cristae, and that the strength of these stimuli is varied by mechanical influences that reach the cristae, when movements, i.e. accelerations of the head, take place.

Chemistry. — "*The Metastability of the Elements and Chemical Compounds in a Consequence of Enantiotropy or Monotropy.* VI. By Prof. ERNST COHEN, Dr. W. D. HELDERMAN and Dr. A. L. TH. MOESVELD..

(Communicated at the meeting of January 29, 1924).

Introduction.

As a result of our investigations, communicated under the above title [1]), it can be deduced, that the solid substances, which we daily handle, are often metastable mixtures of the modifications in which those substances can exist. Accordingly the thermochemical "Constants" of these substances will be quite accidental quantities, functions of the previous history of the material.

While we have hitherto proved the existence of such metastable mixtures in many cases by pyknometric, potentiometric and Röntgen-ray methods, other expedients can obviously be adopted for this purpose in certain cases.

For example, the heat of solution (we mean here by "Heat of solution" the so-called heat of solution in a very large mass of the solvent) of an element or of a compound respectively will be a function of the masses of the $\alpha, \beta, \gamma, \ldots$ modifications which are present in the particular object under thermochemical investigation. Since further the transformation of the metastable forms $\beta, \gamma \ldots$ into the stable α-form always takes place with evolution of heat, the heats of solution which we know up to now, will be found to be numerically smaller (or larger respectively), than those of the stable α-modifications of the substances under investigation, according as these exhibit a negative (or respectively a positive) heat of solution.

Experimental.

1. Bearing in mind what has been said above, in the investigation described here we test the case of cadmium iodide. In a previously communicated paper [2]) it was shown by the pyknometric method

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[1]) Zeitschr. f. physik. Chemie **94**, 450, 465, 471 (1920).
These Proceedings **18**, 961 (1915).

[2]) Zeitschr. f. physik. Chemie **94**, 471 (1920).

that this salt can appear in two modifications, which exhibit the mutual relation of monotropy. The salt which is obtained, for example, by concentrating an aqueous solution is always a metastable mixture of these two modifications in arbitrary proportions.

It was also shown there that by evaporating a methyl alcohol solution at room temperature, or by subliming the solid salt, preparations resulted which contained the metastable modification in a greater proportion.

If the metastable mixture was heated to about 150° for some days the pure stable modification was obtained, the density of which was 5.670 at 30° C. The density of the metastable form is much less than that of the stable, so that for example the metastable mixtures investigated by us, exhibited densities (at the same temperature) which varied between 5.211 and 5.65.

2. Then in order to establish the occurrence of a metastable mixture of cadmium iodide in the calorimetric way, we have determined both the heat of solution of the stable (α) cadmium iodide, and also that of a metastable mixture. We obtained the last by the rapid evaporation of a methyl alcohol solution at about 30° C. In both cases the final concentration in the determination of the heat of solution corresponded to $CdI_2 . 400\ H_2O$.

Although it would actually have sufficed for our purpose, to show that the stable modification and the metastable mixture of α and β cadmium iodide exhibited a clearly defined difference in their heats of solution, we carried out the measurements in a strictly quantitative manner throughout; we previously determined the specific heat of the final solution, which enters into the calculation of the calorific value in absolute units.

Meanwhile we have also used this opportunity of determining the specific heat of a number of cadmium iodide solutions of different concentrations. By the method of least squares an equation was calculated from the results obtained thus, which gives the specific heat of the solutions in question as a function of their concentration. We will describe later in greater detail this investigation, in which we used the electrical adiabatic calorimeter which we had constructed previously[1]); it is sufficient to note here that the specific heat of the solution $CdI_2 . 400\ H_2O$ has the value 0.9571 at 18.4° C.

3. In order to determine the different heats of solution which are in question here, we used an electrical adiabatic calorimeter

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[1]) Zeitschr. f. physik. Chemie 95, 305 (1920).

which we had constructed especially for such measurements. We hope to give a description of this apparatus shortly. We only need note here that we tested its practicability and accuracy by using it to determine the heat of solution of KCl (final concentration $KCl.200H_2O$).

We had already accurately determined this quantity of heat[1]) and were therefore in a position to compare the value found with the new apparatus with the older one. Table I gives the experimental results.

TABLE I.

Heat of solution of KCl to $KCl.200H_2O$ at 18°.4.

Weight of water in gms.	Weight of KCl in gms.	Spec. heat of solution.	Heat of solution at 18°.4.	Heat of solution at 18°.4 Mean.
1100.00	nil	0.9992	101.50 water-equiv.	101.24 (water-equiv)
1100.00	nil	0.9992	100.98	
1103.96	22.7915	0.9722	— 4399	—4400
1104.06	22.7930	0.9722	— 4402	

While we found the heat of solution with the new apparatus to be — 4400 gm. calories, our earlier measurements (at 18°) gave — 4397 gm. calories, in good agreement with this. Hence the practicability and accuracy of the apparatus has been checked satisfactorily. Since we had meanwhile made some changes in the apparatus before we carried out the experiments with cadmium iodide, we determined the water-equivalent of the calorimeter anew. We found then the values 121.74 and 120.88 respectively, mean value 121.3.

4. The stable *α*-cadmium iodide, whose heat of solution we investigated, was prepared in the following manner: turnings of pure cadmium ("Kahlbaum") which only contained 0.005 per cent. of impurity, were agitated for some hours at ordinary room-temperature in a shaking-machine, with resublimed iodine and distilled water, in a large wide-necked flask. The excess of metal still present as well as the cadmium hydroxide produced in the reaction were removed by filtration. Then the clear solution is evaporated to dryness in an Ostwald burner. The mass of crystals was separated from

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[1]) **Zeitschr. f. physik. Chemie 96, 259 (1920).**

adhering liquid on a Buchner funnel. It was washed with a small quantity of water, and dried over sulphuric acid in vacuo. The salt was occasionally ground up in an agate mortar so as to open up the vacuoles as much as possible.

In this way we obtained three preparations, whose densities were as follows. (Duplicate estimations were always made which differed from one another at the most by three units in the third decimal place:[1])

$$D_{4^\circ}^{30^\circ.00} = 5.657,\ 5.659,\ 5.650.$$

Since according to our earlier investigations the stable (α) modification shows a density of 5.670 at this temperature, the deduction must be drawn from the results given here that the three preparations still contained a certain proportion of the metastable (β) modification. In order to obtain complete stabilisation we adopted the following method:

A part of the salt was heated for about 48 hours in contact with its saturated solution at approximately 80°. After the solution had been removed, this treatment was followed by the heating of the salt to about 150° for two to three days and nights. Complete stabilisation was generally achieved after this treatment. We found:

$$D_{4^\circ}^{30^\circ.00} = 5.667 \text{ and } 5.669$$

respectively.

Approximately equal quantities of these preparations were mixed and used for the calorimetric determinations.

5. In order to obtain a preparation which contained a greater proportion of the *metastable* modification, we proceeded in the following manner: cadmium iodide was dissolved in pure methyl alcohol at the ordinary temperature so as to give a saturated solution. This solution was evaporated as rapidly as possible at about 30°, by aspirating dry air through the solution by means of a water-pump. The residue was dried over sulphuric acid in vacuo, care being taken to grind it up frequently in a mortar during the process. A preparation was obtained in this way which gave a density $D_{4^\circ}^{30^\circ.00} = 5.305$.

This preparation was used without further treatment for the calorimetric measurements.

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[1] For the experimental methods refer to Zeitschr. f. physik. Chemie, **94**, 471 (1920).

6. Table 2 gives the experimental results.

TABLE II.

Weight of water in gms.	Weight salt in gms.	$D_{4^\circ}^{30^\circ.00}$ of the salt.		Spec. heat of the solution	Heat of solution at 18°.2 in gm. calories.		Heat of solution at 18°.2 in gm. calories. Mean.
1099.78	55.8706	5.668	stable	0.9571	— 981	stable	— 976
1097.32	55.7453	5.668		0.9571	— 971		
1098.74	55.8133	5.668		0.9571	— 976		
1100.00	55.8791	5.302	meta-stable	0.9571	— 929	meta-stable	— 927
1101.07	55.9861	5.302		0.9571	— 922		
1100.00	55.8795	5.302		0.9571	— 930		

Thus whilst the heat of solution of the *stable* (α) modification is found to be —976 gm. calories at 18·2° C., that of the *metastable* mixture is —927 gm. calories. There is therefore a difference of no less than 5 per cent.

Corresponding to the positive heat of stabilisation the heat of solution of the stable salt has the higher numerical value.

7. After this investigation we determined again the density of the unused portion of our metastable preparation and found $D_{4^\circ}^{30^\circ.00} = 5.302$. Accordingly it had not appreciably altered since its preparation two months before, while preserved in a completely dry condition.

8. For a closer check of the results obtained up to this point we proceeded in the following manner: the metastable preparation was heated in its saturated solution to 80°, and later somewhat further stabilised by drying at 150°, though not completely (since the heating had been interrupted) as the density $D_{4^\circ}^{30^\circ.00}$ had risen to 5.596. Then the heat of solution of this new preparation was determined. Table 3 gives the experimental results.

TABLE III.

Weight of water gm.	Weight of salt gm.	$D_{4^\circ}^{30^\circ.00}$ of the salt.	Spec. Heat of solution.	Heat of solution at 18°.2 gm. calories.
1099.62	55.8635	5.596	0.9571	— 951

As was to be expected, the heat of solution lies between the figures previously obtained, of — 927 and — 976 gm. calories.

We must emphasize here that the weight of the salt dried at 150° had not altered, so that enclosed impurities (methyl alcohol) were not present.

9. Only two determinations of the heat of solution of cadmium iodide are to be found in the literature. One is due to JUL. THOMSEN ($t = 19°$ C.). In his experiments [1]) the final concentration also corresponded to $CdI_2 . 400\ H_2O$. He found the values — 943 and — 981 gm. calories respectively. As a mean value he gave the figure —960 gm. calories.

As in THOMSEN's work detail information as to the previous history of the preparations used is not available, we are unable to tell whether the large difference of 4 per cent between the two individual determinations is to be ascribed to the previous history of the material or to other causes.

The second series of determinations is due to H. S. TAYLOR and G. ST. JOHN PERROT [2]), who brought the iodide into solution in 0.5 N. and 1 N. potassium iodide solutions. Here also the same comment applies.

10. Our own investigations on the polymorphy of elements and compounds, as well as these that have been carried out by other inquirers in recent years, have shown that precisely those methods of treatment which play an important part in the purification of materials (crystallization, sublimation, electrolysis, melting and the accompanying cooling process) are favourable for the production of the *metastable* modification. If there is a retardation of the stabilisation of the modification formed, or if nuclei of different modifications have been formed during the cooling, which afterwards grow, this results in the production of a metastable mixture, as is the case with cadmium iodide.

It appears to us a remarkable discovery that over half-a-century ago phenomena of this type had been observed by PERSON [3]) and later also by BERTHELOT [4]) and studied quantitatively by calorimetric methods. Apparently this work has escaped the notice of chemists and physicists.

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[1]) Journ. f. prakt. Chemie (2) 16, 328 (1877); Thermochemische Untersuchungen, Leipzig 1883. Bd. 3, 185, 201; Systematische Durchführung thermochemischer Untersuchungen. Stuttgart 1906. p. 250.

[2]) Journ. Amer. chem. Soc. 43, 484 (1921).

[3]) Ann. de chim. et de phys. (3) 33, 452 (1851).

[4]) Ann. de chim. et de phys. (5) 29, 198, 231, 239, 295 (1883).

11. Person found that the heat of solution of sodium chloride in water is less negative by 90 gm. calories per equivalent of NaCl when one uses for the experiment salt cooled after preliminary fusion, than when the salt is not previously fused. Berthelot established this fact not only for NaCl (the specimen studied by him have a difference of 180 gm. calories per equivalent) but observed smaller phenomena in all (14) materials studied by him in this manner. He showed thus that the salt which had been fused before the experiment always gave a less strongly negative heat of solution than that which had not had this previous treatment. In addition he found that the difference in question decreased in course of time, so that the heat of solution was again the same after some months, if the material which had been fused previously were kept during this time at ordinary room-temperature.

Very striking, and in complete agreement with our earlier investigations, is the fact that the difference decreases if the salt is ground up after solidification and before the calorimetric determination.

Table 4 gives the experimental results.

TABLE IV.

Name of salt.	Heat of solution of previously fused salt — heat of solution of salt which had not been fused gm. calories per equivalent.		
	After fusing.	After 10 days.	After 2 months.
KCl	+ 170	+ 80	+ 20
KBr	+ 210		—
KI	+ 90		—
NaCl	+ 180		+ 10
$BaCl_2$	+ 30		—
$SrCl_2$	+ 80		—
$CaCl_2$	+ 150		—
$HgCl_2$	+ 400		—
$HgBr_2$	+ 300		—
HgI_2	+1500		—
K_2SO_4	+ 150		0
Na_2SO_4	+ 470		+ 260
K_2CO_3	+ 170		+ 70
Na_2CO_3	+ 350		+ 280

In connection with this table it should be noted that the accuracy of the determination was ± 10 gm. calories per equivalent. The possible error in the differences in the second column is therefore around 20 gm. calories per equivalent; with the mercury salts investigated about 40 gm. calories.

12. While **Berthelot** discusses at first the possible causes of this phenomenon in a very tentative manner[1]): "De telles différences représentent sans doute certains états transitoires, dérivés de la constitution physique du liquide, et que l'on traduisait autrefois par le mot de *chaleur de fusion retenue dans l'état solide*", he appears later[2]) to have formed a clear conception. After he had become aware of the phenomenon in the case of HgI_2, he writes: "Ce dernier écart répond à une différence d'état moléculaire tellement tranchée que les deux corps sont regardés comme isomères. Mais la même interprétation paraît applicable à tous les cas analogues".

13. If Table 4 be examined more closely at the present day, the striking fact appears that, since **Person** and **Berthelot** published their researches, it has been discovered that no less than fifty per cent. of the substances investigated are able to occur in several modifications, ($BaCl_2$, $HgCl_2$, $HgBr_2$, K_2SO_4, Na_2SO_4, K_2CO_3), while it is continually becoming established more certainly that polymorphy is a general property of solid substances.

From the view-point of our previous communications and of the present paper it would appear to be extraordinarely likely that the substances given in Table 4 in connection with which polymorphy has not been proved up to now, will show this property.

We hope to return to this subject shortly.

SUMMARY.

By calorimetric methods a proof is given that the thermochemical constants of cadmium iodide which we know up to the present cannot be relied upon, since they refer to an undefined mixture of the α and β cadmium iodide. Further attention is directed to the quantitative investigations carried out half-a-century ago by **Person** and later by **Berthelot**, which established that the thermochemical constants of solid substances are a function of their previous history.

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[1]) Ann. de chim. et de phys. (5) 29, 198 (1883); especially p. 217.

[2]) Ann. de chim. et de phys. (5) 29, 295 (1883); especially p. 309 and 310.

Confidence can only be put in those constants which were determined with the *pure stable*, or *metastable* modification respectively, of the substance.

It is desirable that such measurements should be carried out.

We wish to express our best thanks to Mr. H. G. S. Snijder, who very kindly assisted us in the work described here.

Utrecht, Jan. 1924. van 't Hoff-*Laboratorium.*

Chemistry. — *"The Metastability of the Elements and Chemical Compounds in a Consequence of Enantiotropy or Monotropy".* VII. By Prof. ERNST COHEN.

(Communicated at the meeting of January 29, 1924).

Historical Note.

1. In connection with my researches on the metastability of Tin (tin disease) I have collected a number of papers, met with amongst the writings of the classical authors, which show that this phenomenon was already known in early times [1].

In my first communication [2] on the subject indicated in the title of this paper I directed attention to the fact that, on the occasion of a winter spent on the Kara Sea, some investigations were carried out at VAN 'T HOFF's suggestion by RUYS [3] on the velocity with which monoclinic sulphur becomes stabilised at low temperatures. In these it was established that molten sulphur, on sudden cooling, only changes in small proportion immediately into the rhombic modification. At low temperatures (— 36° to — 15°; — 31° to — 8°) the two modifications could exist in contact with one another for fairly long times (12 and 10 days respectively), and even at ordinary room-temperature it was a long time before the stabilising process had come completely to an end.

2. But observations relative to this phenomenon are to be found in the literature of much older date, and which extend over a much longer time, that prove that in the case of sulphur equilibrium is not even reached after several years. Accordingly the sulphur which remains after the melting process is a metastable mixture, whenever special precautions have not been taken.

In a communication with the date 1847 the geologist CHARLES SAINTE CLAIRE DEVILLE [4] directed attention to this phenomenon. He writes:

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[1] Zeitschr. f. physik. Chemie 36, 513 (1901).

[2] Verslagen Kon. Akad. v. Wetenschappen te Amsterdam, 24, 886, (1915).

[3] Rec. des Trav. Chim. des Pays-Bas 3, 1 (1884); Zeitschr. f. Kristallographie 8, 593 (1884).

[4] C. R. Paris 25, 857 (1847).

„Voilà quel est le mouvement observé dans les densités $\left(d\frac{t^\circ}{4^\circ}\right)$ pendant cette transformation:

26 avril **1845**	1.9578, aussitôt après la fusion.
26 avril	2.0022, trois heures après.
27 avril	2.0055
28 avril	2.0131
29 avril	2.0192
1er mai	2.0307
3 mai	2.0348
3 mai **1846**	2.0453
9 juin	2.0440
4 décembre **1847**	2.0498

On voit que, depuis dix-huit mois, l'accroissement de densité, s'il a lieu, se fait bien lentement. Un soufre cristallisé à chaud en 1839, et qui m'a été remis par MM. Favre et Silbermann, ne présentait, en janvier 1847, qu'une densité de 2.0508".

3. In this connection it should be noted that the density of pure octohedral sulphur (stable at ordinary temperatures) is about 2.07. A more precise value cannot be given at the moment, since the phenomena discussed here have not been sufficiently closely studied up to the present.

4. From the evidence given here sufficient proof has been established that it was already known more than seventy-five years ago that molten sulphur, when cooled without special precautions, yields a metastable mixture of several different modifications.

Utrecht, January 1924. VAN 'T HOFF-*Laboratorium.*

Physiology. — *"Concerning the Influence of the Administration of Iron on the Respiration of the Red Bloodcorpuscles."* By Prof. A. A. Hijmans van den Bergh and M. J. Roessingh."

(Communicated at the meeting of December 27, 1923).

It is well-known that the haemoglobin-containing red bloodcorpuscles are the carriers of the oxygen of the atmosphere, required for the oxidative processes of the tissue cells. Morawitz was the first to ascertain whether the red bloodcorpuscles have a respiration of their own. The author demonstrated that in man the oxygen-consumption of the red bloodcorpuscles is, under normal conditions, very trifling, amounting to some 3 or 4 °/₀. This lack of respiration is in keeping with other properties of these cells: absence of a nucleus, absence of the function of reproduction, which allot to them a place different from that of all living cells and induce us to consider them as an intermediate form between dead and living matter.

Contrary to what we observe in man and in mammals the oxygen-consumption of the red bloodcorpuscles in birds is of considerable importance. The difference is apparently due to the human erythrocytes being non-nucleated, whereas the red cells in birds, as is well-known, contain a nucleus. That, however, this view is not quite correct, is borne out by further experiments made by Morawitz in collaboration with Masing. The oxygen-consumption of the red bloodcorpuscles of mammalia is indeed largely increased through withdrawal of blood (bloodletting), and in general through all sorts of operations (administration of blooddestroying poisons, such as phenylhydrazin) which make the animal anaemic and bring about a strong regeneration of young bloodcorpuscles. In these cases, however, no nucleated red cells will enter the peripheral blood, or, if they do, their number is so small, that they cannot be responsible for the increase of the oxygen consumption. It is generally received that the increased oxygen consumption in these circumstances is owing to the youthful state of the red cells. Probably the oxidative processes are related to the presence of nuclear fragments in these cells, although they are not demonstrable under the microscope, or to their chemical composition, which is supposed to be different in the young and the old corpuscles. Just as in the animal experiment the human blood also appears to have an increased oxygen consumption, whenever the bloodbuilding organs are in a state of active bloodformation. Thus we find an increased O consumption in pernicious anaemia, and in the blood of individuals regenerating actively, as in all sorts of intoxications and after hemorrhage, and in general with excessive activity of the bone marrow.

On a previous occasion Dr. ROESSINGH has in my laboratory tried to ascertain by this method whether the anaemia that is often met with in carcinoma, occurs with an active regeneration of red blood-corpuscles.

The result was negative: The O consumption of the blood of individuals with carcinoma was not greater than in normal men. On the other hand ROESSINGH found in pernicious anaemia, and in the blood of individuals suffering from other forms of anaemia, and also shortly after bloodtransfusion, considerable increase (instead of the normal value 3—4 °/₀ : some 10 to 60 °/₀).

By means of the determination of the O consumption of the red blood cells ROESSINGH has recently studied the effect of iron on bloodformation.

Anaemic patients were given large doses of iron during several weeks. Before and after treatment the oxygen consumption of the red blood-corpuscles was measured.

The defibrinated blood is centrifugalized; then the red blood-corpuscles are washed out with a physiological NaCl-solution in order to remove the serum. Now the emulsion of blood-corpuscles is shaken up with air for saturation of the haemoglobin with oxygen. Particular attention should be given to absolute sterility, because many bacteria have a considerable consumption of oxygen, which, if disregarded, must lead to grave errors. After the haemoglobin has been saturated with O, the quantity of loosely combined oxygen in a portion of the blood is determined after the method of HALDANE in a BARCROFT apparatus. Another portion of the blood is placed for four hours in the incubator at 37°, being well shut off from the air; after this a determination of the loosely bound oxygen in the second blood-portion is made. The difference in the amount of the loosely bound oxygen before and after the incubation represents the oxygen withdrawn by the corpuscles from their own haemoglobin for their own respiratory function.

The results of this inquiry have been tabulated below. The different blood samples apparently behave in a different way, which we have not been able to account for. In one case the O.-consumption after Fe-administration appeared to be diminished, which is most likely due to an experimental error. In another experiment it remained constant, but in the majority of cases a considerable augmentation was noted, in consequence of, or anyhow subsequent to the Fe-administration. In these cases we are inclined to conclude that the iron has stimulated the bonemarrow to an active formation of young cells.

However, we have to discuss an objection. It is well-known that in many biochemical processes iron is believed to play the part of a catalyst. It might be asked whether the iron may perhaps be combined in the molecule of the corpuscles, thus inciting respiration. This hypothesis deserves the more consideration in view of WARBURG's observation that the sea-urchin eggs consume more oxygen and give off more CO_2 after the addition of highly diluted ferro-, or ferri-salts (60—100 $^0/_0$) than without this addition. In order to test this objection Dr. ROESSINGH has added dilute iron-salts to emulsions of red blood-corpuscles and examined the O-consumption before and after the addition. The Fe-salts proved to exert no influence.

For the present we may therefore conclude that in all probability large doses of iron administered to anaemic patients stimulate the bloodbuilding organs, so as to enable them to supply the peripheral blood with young red bloodcells.

We are tempted to compare the oxygen consumption of the red bloodcorpuscles with glycolysis. This process, thoroughly studied by LÉPINE, has occupied the attention of a great number of researchers. Many have called in question LÉPINE's conclusions or have denied their great importance, so that the subject was dropped. Not until quite recently, it has again been studied with much interest.

We wanted to know whether glycolysis, by which is meant the spontaneous disappearance of glucose from the blood, has anything to do with the internal respiration of the red bloodcorpuscles. When studying this problem in my laboratory, Dr. VAN STEENIS found, contrary to the current opinion, that the red bloodcorpuscles in man possess a glycolytic function. We then ascertained whether the presence of oxygen has a considerable influence upon this process. To this end two series of experiments were compared. In one series the fluid was shut off from the air. In the other we allowed oxygen to pass through the fluid. We found no differences. Ultimately we compared glycolysis of bird's blood and human blood. If there were any relation between glycolysis and the oxygen-consumption of the red bloodcorpuscles a higher degree of glycolysis might be expected in bird's blood than in human blood, since indeed the internal respiration in birds is much more considerable than in man. Such a difference we failed to detect. We, therefore, feel justified in concluding that the two processes are not related.

	Before the admin. of iron			*After* the admin. of iron		
	Haemogl. content	Red blood-corpuscl.	Oxygen consump.	Haemogl. content.	Red blood-corpuscl.	Oxygen consump
1. *R.* Chron. nephritis. 30 days; 2.25 gr. Fe daily			14.56 %			72.04 %
2. *Miss V.* Anaemia after hemorrhage. 20 days; 2.25 gr. Fe daily	70 %	4.400000	none			13.10 %
3. *v. d. H.* Tuberc. pulmonum. 30 days; 2.25 gr. Fe daily			45.66 %			41 %
4. *Miss D.* Chron.nephritis. 14 days; 2.25 gr. Fe daily	42 %	1.860000	none	52 %	2.200000	46.33 %
5. *Miss deL.* Splenomegaly. 30 days; 2.25 gr. Fe daily	105 %	4.000000	2.44 %	100 %	4.400000	51.28 %
6. *Miss S.* Tumor mediastini. 18 days; 2.25 gr. Fe daily	90 %	4.200000	11.30 %	85 %	3.800000	28.00 %
7. *Miss Z.* Purpura hemorrhage. 18 days; 2.25 gr. Fe daily	70 %	3.600000	none	85 %	3.850000	57.72 %
8. *v. E.* Malignant abdominal tumor. 18 days; 2.25 gr. Fe daily	55 %	3.800000	12.67 %	50 %	4.440000	22.40 %
9. *Miss v. d. H.* Adipositas. 25 days; 2.25 gr. Fe daily	48 %	3.500000	none	64 %	3.800000	16.27 %
10. *Miss v. D.* Tuberc. peritonei. 46 days; 2.25 gr. Fe daily	64 %	4.200000	none	82 %	4.510000	3.73 %
11. *Miss D.* Colitis. 22 days; 200 mgr. Fe daily	48 %	3.180000	5.54 %	55 %	4.200000	38.97 %
12. *H.* Sacroma lymphatic glands 21 days; 2.25 gr. Fe daily	100 %	5.120000	4.27 %	98 %		2.12 %

Continued.

	Before the admin. of iron.			*After* the admin. of iron.		
	Haemogl. content.	Red blood-corp.	Oxygen consump.	Haemogl. content.	Red blood corp.	Oxygen consump.
13. *de B.* Tuberc. abscess. 21 days; 2.25 gr. Fe daily	85 %	4.900000	none	85 %	4.780000	7.63 %
14. *B.* Otitis chronica. 21 days; 2.25 gr. Fe daily	100 %	5.170000	none	95 %	5.000000	none
15. *v. Z.* Pernic. anaemia. 21 days; 2.25 gr. Fe daily	48 %	1.370000	none	45 %	1.530000	10.09 %
16. *L.* Carcin. ventriculi. 14 days; 2.25 gr. Fe daily	31 %	3.170000	none	32 %	2.360000	none
17. *Miss B.* Anaemia. 19 days; 2.25 gr. Fe daily	43 %	3.120000	44 %	60 %	4 400000	none
18. *Miss C.* Spondylitis. 21 days; 2.25 gr. Fe daily	100 %		none	100 %		none
19. *W.* Anaemia. 25 days; 2.25 gr. Fe daily	50 %	4.400000	none	62 %	5.040000	52.13 %
20. *Miss V.* Arteriosclerosis. 21 days; 300 mgr. Fe daily	95 %		4.21 %	95 %		none
21. *Miss G.* Apoplexy. 21 days; 2.25 gr. Fe daily	95 %		none	100 %		22.25 %
22. *Miss V.* Diabetes. 21 days; 1.5 gr. Fe daily	88 %	4.530000	21.31 %			23.82 %
23. *Miss E.* Anaemia. 23 days; 2.25 gr. Fe daily	54 %	4.400000	11.7 %	88 %	4.250000	27.25 %

Geophysics. — *"On the flattening and the constitution of the earth".* By Prof. W. DE SITTER.

(Communicated at the meeting of March 29, 1924).

1. The outer potential of any body, which is symmetrical with reference to an equatorial plane and an axis perpendicular to this plane, may be developed in a series, which, to the order of accuracy here required is

$$V = \frac{fM_1}{r}\left[1 - \frac{2}{3}\frac{Jb^2}{r^2}P_2(\sin\delta) + \frac{4}{15}\frac{Kb^4}{r^4}P_4(\sin\delta)\right], \quad . \quad . \quad . \quad (1)$$

M_1 being the total mass, f the constant of gravitation, and b the equatorial radius. P_2 and P_4 are spherical harmonics of the declination δ above the equatorial plane and J and K are constants characteristic of the body. We have

$$J = \frac{3}{2}\frac{C-A}{M_1 b^2},$$

C en A being the moments of inertia with respect to the polar and an equatorial diameter respectively. If we introduce the ratio

$$H = \frac{C-A}{C},$$

we have

$$J = qH \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

with

$$q = \frac{3}{2}\frac{C}{M_1 b^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If the surface of the body is an equipotential surface, its equation may be written as that of a spheroid

$$r = b\left[1 - \varepsilon \sin^2\varphi' - \left(\frac{3}{8}\varepsilon^2 + \varkappa\right)\sin^2 2\varphi'\right], \quad . \quad . \quad . \quad (4)$$

φ' being the geocentric latitude and ε the flattening.

This spheroid deviates from an ellipsoid of rotation by a depression $-b\varkappa\sin^2 2\varphi'$ reaching its maximum at the latitude 45°.

If the body rotates with a constant angular velocity ω, so that at its surface the potential is (V_1 being the value of V for the surface, $\varphi' = \delta$):

$$V_1 + \frac{1}{2} r^2 \omega^2 \cos^2 \varphi',$$

then the conditions that the surface shall be an equipotential surface are

$$\varepsilon - J = \frac{1}{2} \varrho_1 + \frac{1}{2} \varepsilon^2 - \frac{1}{7} \varepsilon \varrho - \frac{4}{7} \varkappa = \cdot 0017287. \quad . \quad . \quad (5)$$

$$K = \frac{24}{7} \varkappa + 3\, \varepsilon^2 - \frac{15}{7} \varepsilon \varrho = \cdot 0000109.$$

where

$$\varrho_1 = \frac{\omega^2 r_1^3}{f M_1}$$

is determined by

$$\varrho_1 + \frac{2}{3} \varrho_1^2 = \frac{\omega^2 r_1}{g_1},$$

r_1 being the mean radius, and g_1 the acceleration of gravity at the latitude of this mean radius. This gives

$$\varrho_1 = \cdot 00344992 \pm \cdot 00000002.$$

In the small terms in (5) and (6) I have omitted the index of ϱ_1, and I have adopted

$$\varepsilon = {}^1/296{\cdot}5 = \cdot 003373$$
$$\varkappa = \cdot 00000050.$$

The value of $\varkappa$ is, of course, entirely unknown. If the earth were homogeneous, it would be zero, on the other hand it cannot exceed $\frac{5}{16} \varepsilon \varrho - \frac{1}{4} \varepsilon^2 = \cdot 00000080$, so that the adopted value appears plausible. It corresponds to 3.2 meters, and is thus entirely irrelevant.

The equation (5) is independent of any assumption regarding the distribution of mass inside the earth. For q on the other hand we have no rigorous equation of this kind, but the theory of CLAIRAUT on the constitution of the earth enables us to derive a very approximate value of q.

If we wish to go beyond an accuracy of the order of a unit in the denominator of ε, corresponding to the fifth decimal place in ε itself, it is necessary to include the second order. On the other hand, if the third order is neglected, all figures beyond the seventh decimal are meaningless. The theory of CLAIRAUT has been developed to the second order by DARWIN[1]), and others. By taking as independent variable the *mean*, instead of the equatorial, radius, the formulas become somewhat simpler, and at the same time the range of uncertainty of q is considerably narrowed. As this change of independent variable does not affect the essential parts of the theory, which are well known, I will only state the principal steps and formulas very succinctly, without going into the details of their derivation. As we require only one term beyond the one of the lowest order in any equation, we can choose at random any one of the several definitions of the mean radius, which are equivalent to the first order. We may suppose it to be the radius of the sphere of equal volume.

2. On the theory of CLAIRAUT the surfaces of equal density are equipotential surfaces. Let β be the mean radius of any such surface, expressed in that of the outer surface as unit, then the equation of this surface becomes

$$r = \beta \left[1 - \frac{2}{3} \left(\varepsilon' + \frac{2}{3} \varepsilon^2 \right) P_2 (sin\, \varphi') + \,..\right], \quad . \quad . \quad (7)^{2)}$$

P_2 being again the spherical harmonic. The harmonic of the fourth order is not needed for our immediate purpose, and has been omitted from the formulas. We have put

$$\varepsilon' = \varepsilon - \frac{5}{42} \varepsilon^2 + \frac{4}{7} \varkappa \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

In the terms of the second order it is not necessary to distinguish between ε and ε', and the accent is dropped.

The potential V at any point within the earth, of which the coordinates are r and φ', is given by

$$\frac{V}{fW} \cdot r = D \left[1 + \frac{1}{2} \varrho \frac{r^3}{\beta^3} cos^2 \varphi' \right] - \frac{2}{5} \left[S \frac{\beta^5}{r^2} + T \frac{r^3}{\beta^3} \right] P_2\, sin\, \varphi' + \ldots \quad (9)$$

where

$$\varrho = \frac{\omega^2 \beta^3}{fM} = \frac{\varrho_1}{D},$$

W being the volume, and M the mass within the surface of which the mean radius in β, and

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[1]) Scientific papers III, p. 78 = M.N. 60, p. 82 (1900).

[2]) The numbering of the formulas is the same as in the somewhat more extended publication in B. A. N. 55, which explains why some numbers are missing here.

$$D = \frac{3}{\beta^3} \int_0^\beta \delta \beta^2 d\beta,$$

$$S = \frac{1}{\beta^5} \int_0^\beta \delta \frac{d}{d\beta} \left[\beta^5 \left(\varepsilon' + \frac{2}{7} \varepsilon^2 \right) \right] d\beta,$$

$$T = \int_\beta^1 \delta \frac{d}{d\beta} \left[\varepsilon' + \frac{16}{21} \varepsilon^2 \right] d\beta.$$

In these formulas δ is the density, expressed in the mean density as a unit, and consequently D is the mean density within the surface β, expressed in the same unit [1]. For the outer surface we have of course

$$D_1 = 1 \quad , \quad S_1 = \frac{5}{3} J \quad , \quad T_1 = 0.$$

The comparison of (7) and (9) gives for the condition that (7) shall be an equipotential surface:

$$D\left(\varepsilon' + \frac{2}{7} \varepsilon^2 - \frac{1}{2} \varrho \right) - \frac{3}{5} (S + T) = \frac{4}{21} \varepsilon (\varrho_1 - 3T). \quad . \quad (10)$$

The right hand member of (10), which is of the second order, has been simplified by means of the left hand member equated to zero.

Putting further

$$\zeta = -\frac{\beta}{D} \frac{dD}{d\beta} = \frac{\beta}{\varrho} \frac{d\varrho}{d\beta},$$

we have

$$\zeta = 3 \left(1 - \frac{\delta}{D} \right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

This equation is now *rigorous* in consequence of the introduction of the mean, instead of the equatorial radius as argument.

Differentiating (10), and putting

$$\eta' = \frac{\beta}{\varepsilon'} \frac{d\varepsilon'}{d\beta},$$

we find, after reduction

$$\eta' \left[D\left(1 + \frac{4}{7} \varepsilon \right) - \frac{4}{21} \varrho_1 + \frac{4}{7} T \right] = 3 D \left(1 + \frac{2}{7} \varepsilon \right) - 3 \frac{S}{\varepsilon'}, \quad . \quad (13)$$

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[1] It should be remarked that, adopting these units, we have $M_1 = \frac{4}{3} \pi r_1^3 D_1 = \frac{4}{3} \pi$.

and differentiating again, we find the differential equation for η:

$$\beta\frac{d\eta'}{d\beta} + \eta'^2 + 5\,\eta' - 2\,\zeta\,(1+\eta') - \frac{4}{21}\,\zeta\, Q = 0 \quad .\;\;.\;\;. \quad (14)$$

where

$$Q = 7\,\varrho\,(1+\eta') - 3\,\varepsilon\,(1+\eta')^2 - 4\,\varepsilon. \quad .\;\;.\;\;.\;\;.\;\;. \quad (15)$$

Omitting the term with Q (14) is the well known differential equation correct to the first order. The introduction of ε' for ε has removed from (14) a term of the form $\varepsilon\eta^2$, thus having all small terms multiplied by ζ. DARWIN, using the equatorial instead of the mean radius, finds the same equation, but in his value of Q the numerical coefficients of the second and third term are larger than here.

From (14) we derive, as was first done by RADAU:

$$D\beta^5\sqrt{1+\eta'} = 5\int_0^\beta D\beta^4\,F(\eta')\,d\beta, \quad .\;\;.\;\;.\;\;.\;\;. \quad (16)$$

where

$$F(\eta') = \frac{1 + \frac{1}{2}\,\eta' - \frac{1}{10}\,\eta'^2 + \frac{2}{105}\,\zeta\,Q}{\sqrt{1+\eta'}} \quad .\;\;.\;\;.\;\;.\;\;. \quad (17)$$

The value of $F(\eta')$ is always very near to unity. If by $1+\lambda$ we denote a certain average value of it over the range of integration, we have

$$\int_0^\beta D\beta^4\,d\beta = \frac{D\beta^5\sqrt{1+\eta'}}{5\,(1+\lambda)}. \quad .\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;. \quad (18)$$

Now we have, to the required order of accuracy

$$C = \frac{8}{15}\,\pi\int_0^1 \delta\,\frac{d}{d\beta}\left[\beta^5\left(1+\frac{2}{3}\,\varepsilon\right)\right]d\beta = \frac{8}{3}\,\pi\int_0^1 \delta\beta^4\,d\beta + \frac{2}{3}\,(C-A),$$

and since in our units $M_1 b^2 = \frac{4}{3}\,\pi\left(1+\frac{2}{3}\;\right)$, we have by (3)

$$q = 3\left(1-\frac{2}{3}\,\varepsilon\right)\int_0^1 \delta\beta^4\,d\beta + \frac{2}{3}\,J. \quad .\;\;.\;\;.\;\;.\;\;.\;\;. \quad (19)$$

Replacing in the integral δ by its value from (12), then integrating by parts, and using (5), of which only the terms of the lowest order are here needed, we find

$$q = 1 - \frac{1}{3}\varrho_1 - 2\left(1 - \frac{2}{3}\varepsilon\right)\int\limits_0^1 D\beta^4\, d\beta, \quad . \quad . \quad . \quad . \quad (20)$$

or by (18)

$$q = 1 - \frac{1}{3}\varrho_1 - \frac{2}{5}\left(1 - \frac{2}{3}\varepsilon\right)\frac{\sqrt{1+\eta'_1}}{1+\lambda_1}. \quad . \quad . \quad . \quad (21)$$

From the equation (13) we find for the outer surface

$$\varepsilon'\eta'_1 = \frac{5}{2}\varrho_1 - 2\varepsilon' + \frac{10}{21}\varrho^2 + \frac{4}{7}\varepsilon^2 - \frac{6}{7}\varepsilon\varrho, \quad . \quad . \quad . \quad (22)$$

from which, with $\varepsilon = {}^1/_{296\cdot 50} = \cdot 0033727$, corresponding to $\varepsilon' = {}^1/_{296\cdot 60} + \Delta\varepsilon^{-1}$ we find

$$\eta'_1 = \cdot 55893 + \cdot 00863\, \Delta\varepsilon^{-1}.$$

Therefore

$$q = \cdot 50053 - \cdot 00140\, \Delta\varepsilon^{-1} + [\cdot 4983 + \cdot 0014\, \Delta\varepsilon^{-1}]\frac{\lambda_1}{1+\lambda_1} \quad . \quad (23)$$

3. To derive a probable value for λ_1, and to ascertain the limits within which this value may be considered as trustworthy, we must discuss the function $F(\eta')$ given by (17). The first part of this function is the same as discussed by RADAU, POINCARÉ and others, viz.:

$$F_1(\eta) = \frac{1 + \frac{1}{2}\eta - \frac{1}{10}\eta^2}{\sqrt{1+\eta}}.$$

For convenience I drop the accent, which is of no importance here. The function $F_1(\eta) - 1$ is zero for $\eta = 0$, rises to a maximum of $+ .00074$ for $\eta = \frac{1}{3}$, and then decreases again, becoming zero for $\eta = \cdot 528\ (= 5 - 2\sqrt{5})$. For the surface $\eta_1 = \cdot 56$ it is $-.00029$. It is thus positive for practically the whole range, and is larger than $+ .00050$ from $\eta = \cdot 19$ to $\eta = \cdot 44$. Since η increases continually from zero at the centre to $\eta_1 = \cdot 56$ at the surface, we can take as a plausible value

$$F_1(\eta) = 1\cdot 00050 \pm \cdot 00015 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

The "probable error" attached to this value is meant to express the belief that there is an even chance that the true average value of $F_1(\eta)$ is included within a range of .00030, the adopted value 1.00050 being somewhere inside this range, and probably not far from the middle.

To discuss the value of Q we put

$$\varepsilon = \varrho\,(1-v) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (25)$$

We then have

$$Q = Q_1 + Q_2,$$
$$Q_1 = (\eta - 3\eta^2)\,\varrho,$$
$$Q_2 = (7 + 6\eta + \eta^2)\,\varrho v.$$

Consider first the first part Q_1. This varies between the limits $+\cdot 08\,\varrho$, for $\eta = \frac{1}{6}$ and $-\cdot 36\,\varrho$ at the surface. At the centre it is zero. The value of ϱ is always smaller than ϱ_1, $\sqrt{1+\eta}$ is always larger than 1, and it is probable that ζ never considerably exceeds its surface value $\zeta_1 = 1.5$. We can thus take

$$\frac{2}{105}\,\frac{\zeta\varrho}{\sqrt{1+\eta}} < \frac{2}{105}\,\zeta_1\,\varrho_1 = \cdot 00010.$$

Consequently the value of $\frac{2}{105}\frac{\zeta Q_1}{\sqrt{1+\eta}}$ is certainly comprised between the limits $+.000008$ and $-.000036$. We can take as a plausible average

$$F_2(\eta) = -\cdot 000014 \pm .000010 \quad . \; . \; . \; . \; . \quad (26)$$

To discuss the function Q_2 we must consider the possible values of v. Differentiating (25) we find

$$\frac{\beta}{1-v}\,\frac{dv}{d\beta} = \zeta - \eta,$$

from which, since ζ is always larger than η, it follows that v increases continuously from the centre outwards. For the earth the surface value is $v_1 = +\cdot 02$. For v_0 we find upper and lower limits from the formula (42) given by TISSERAND, *Méc. Cél.* II, p. 227, which depend on the density at the centre σ_0. These are:

ρ_0	Limits of $\frac{\nu_0 \rho_0}{\rho_1}$	
1.28	—.25	—.25
1.5	—.26	—.22
2.0	—.33	—.22
2.5	—.39	—.21
3.0	—.42	—.21

We can thus take with certainty

$$-\cdot 40 < \frac{\nu_0\, \varrho_0}{\varrho_1} < -\cdot 21. \quad \ldots\ldots\ldots \quad (27)$$

Now put for abbreviation

$$\frac{Q_2}{\sqrt{1+\eta}} = \omega\, \varrho_1.$$

For the surface we have

$$\omega_1 = +\cdot 17,$$

and for the centre, where $\eta = 0$, $Q_2 = 7 \nu_0 \varrho_0$, we have

$$-2\cdot 8 < \omega_0 < -1\cdot 5,$$

Consequently the product $\zeta\omega$, as we proceed from the surface to the centre, will with some rough approximation follow the diagonal of the following table, starting from the left hand top corner, and reaching the bottom somewhere between the last two colums.

Values of the product $\zeta \,.\, \omega$

ω ζ	.17	0	—.5	—1.0	—1.5	—2.8
1.5	+.25	0	—.75			
1.25	+.21	0	—.62	—1.25		
1.00	+.17	0	—.50	—1.00	—1.50	
.75		0	—.38	— .75	—1.12	—2.10
.50			—.25	— .50	—0.75	—1.40
.25				— .25	—0.38	— .70
.00					.00	.00

It thus appears probable that the average value of this product over the range of integration is included between comparatively narrow limits, say between —.20 and —1.20. We can take as a plausible value

$$\overline{\zeta\omega} = -\cdot 70 \pm \cdot 30,$$

from which

$$F_2(\eta) = \frac{2}{105} \cdot \overline{\zeta\omega} \cdot \varrho_1 = -\cdot 000046 \pm \cdot 000020 \quad . \quad . \quad (28)$$

Adding together the three partial values (24), (26) and (28), we find

$$\lambda_1 = + \cdot 00044 \pm \cdot 00015 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

It will be seen that the effect of the second order term with Q is only about one tenth of the whole.

Substituting this value of λ_1 in (23) we find

$$q = \cdot 50075 \pm \cdot 00008 - \cdot 00140\, \Delta\varepsilon^{-1} + \cdot 499\, \Delta\lambda_1 \quad . \quad . \quad . \quad (30)$$

4. We must now consider the question whether these results, which have been derived from the theory of CLAIRAUT, are applicable to the actual earth.

The actual surface of the earth is, of course, neither a surface of equal density nor an equipotential surface. We can however safely assume that up to a certain distance from the centre the material out of which the earth is made up is — so far as secular forces are concerned — in hydrostatic equilibrium, and consequently satisfies the conditions of the theory of CLAIRAUT. The last surface for which this theory is applicable is called the *isostatic* surface, and will be denoted by S_0. Above this there exist, of course, further equipotential surfaces, but these are not as a rule surfaces of equal density, and not necessarily spheroids. The actual land surface is, of course, not such a surface, but the undisturbed surfaces of the different oceans can be assumed to form parts of one and the same equipotential surface, which is called the *geoid*. This geoid is determined from geodetic measures on the continents, and from determinations of the intensity of gravity on the continents and on the oceans. It is found that it deviates only very little from an ellipsoid of revolution. The ellipsoid of revolution, or rather the spheroid, best fitting the geoid is called the *normal* surface, and is denoted by S. The differences between the geoid and the normal surface never amount to more than a few tens of meters. This fact has led to the well known theory of isostasy, which asserts that within any cylinder erected over a (not too small) surface element ω of the isostatic surface there is the same mass as there would be with a certain ideal distribution, which we can take to be in accordance with the theory of CLAIRAUT. The upper surface that would result if the conditions of this theory were satisfied throughout will be called the *ideal* surface, and will be denoted by S_1.

To this surface S_1 the equations (5) and (2), with the value (21) of q, are applicable. The normal surface S on the other hand is not an equipotential surface, but it is the spheroid best fitting the geoid, which is an equipotential surface. For the condition of the "best fitting" we can take that in the developments of both surfaces in series of spherical harmonics the coefficients of the harmonics of the orders zero, two and four are the same. Then the equation (5) is applicable if for ε we take the compression of the normal surface, and for J its actual value for the real earth. We can, of course, again write down the equation (2), taking for H also its actual value, but now q is determined by (3) and not by (21) and the problem before us is to find the difference between these two values of q.

If the earth were entirely constituted according to the theory of CLAIRAUT, it would be covered by an ocean of an average depth of about 2.4 km., of which the upper surface would be the ideal surface S_1, and the bottom would also be an equipotential surface, which we will call S_b. The true distribution of mass differs from this ideal one on the one hand by an excess of mass in the continents and the shallow seas, and a defect in the deep oceans, and on the other hand by the isostatic compensations of these excesses and defects.

This compensation is assumed to be equally distributed over the layer between the surfaces S_1 and S_b or, in the case of the deep oceans, between S_1 and the bottom of the ocean.

The formulas have been worked out by me in 1915[1]).

Correcting a mistake in the formulas, and treating the layer between S_1 and S_b somewhat more carefully than was done there, we find in units of the seventh decimal place: (see table p. 243).

If there were no isostatic compensation, these numbers would be increased about 55 times. The approximation of the computations is such that each of the partial numbers is correct to a few percents of its amount. The sums may thus easily be a unit or more in error.

We have thus as the result of this computation that the difference

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[1]) These Proceedings, XVII, p. 1295.

$C - C_1$ is entirely negligible, and consequently

$$q = q_1$$

Consequently we can use the value (21) or (30) of q to derive

Parts of the world	$H-H_1$	$\frac{C-C_1}{C}$
1. North Polar regions	+1.7	+ .2
2. Europe	− .1	+ 2.4
3. Asia	+ .8	+14.5
4. North-America	−1.0	+ 5.2
5. Northern Atlantic Ocean	−1.2	− 4.3
6. South-America	+2.3	+ 9.4
7. Southern Atlantic Ocean	+1.2	−10.7
8. Africa	+2.9	+14.3
9. Indian Ocean	+1.2	− 9.2
10. Indian Archipelago	− .7	+ 2.0
11. Australia and New Guinea	+ .5	+ 3.7
12. Pacific Ocean	−9.5	−31.5
13. South Polar regions	−6.6	+ .6
Total	−8.5	− 3.4

ε from H for the actual earth. The difference $q - q_1$ would still be negligible, if there were no isostatic compensation.

For H we find:

$$H - H_1 = - \cdot 0000008,$$

from which

$$\varepsilon - \varepsilon_1 - \cdot 0000004 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (32)$$

This of course is entirely negligible. It means that the polar semidiameter of the ideal surface is 1.8 meters shorter, and the equatorial radius 0.9 meters longer, than of the normal surface. If there were no isostatic compensation however, the difference between ε and ε_1 would be of the order of two units in the denominator.

5. We have from (5) and (2)

$$\varepsilon = \cdot 0017287 + q\,H \quad . \; . \; . \; . \; . \; . \; . \; . \quad (31)$$

The value of H can be derived with great accuracy from the constant of precession. Adopting for the reciprocal of the mass of the moon

$$\mu^{-1} = 81 \cdot 50 \pm \cdot 07 + \Delta \mu^{-1},$$

I find

$$H = \cdot 0032774 + \cdot 0000270 \, \Delta \mu^{-1}.$$

The probable error of H is made up of ± 19 in the seventh decimal place due to the uncertainty of μ, and ± 0.6 due to the constant of precession. Since it has been shown that the value of q derived from the theory of CLAIRAUT may be used for the actual earth, we can substitute (32) into (31). Then taking

$$\varepsilon = \cdot 0033727 + \Delta \varepsilon,$$

we find [1]:

$$597 \, \Delta \varepsilon = - \cdot 0000029 + \cdot 0000135 \, \Delta \mu^{-1} + \cdot 00163 \, \Delta \lambda_1, \quad . \quad (33)$$

from which

$$\frac{1}{\varepsilon} = 296{\cdot}92 \pm {\cdot}136 - 1{\cdot}99 \, \Delta \mu^{-1} - 152 \, \Delta \lambda_1. \quad . \quad . \quad (34)$$

The probable error of $1/\varepsilon$ is made up as follows:

from the precessional constant	± ·004,
from μ	± ·132,
from λ_1	± ·035.

The remaining uncertainty of ε is thus due almost entirely to that of the mass of the moon.

The most important and trustworthy determinations of ε by other methods have already been quoted in my paper of 1915. Shortly after the publication of that paper HELMERT[2]) has published a new determination of ε from the intensity of gravity, which is

$$\varepsilon^{-1} = 296{\cdot}7 \pm 0{\cdot}6$$

The most reliable geodetic determination is that by HAYFORD:

$$\varepsilon^{-1} = 297{\cdot}0 \pm 1{\cdot}2$$

Both agree with (34), but both are very much less accurate. When by the Eros campaign of 1930 the mass of the moon will be better known than it is now, the determination of the compression from the precessional constant will become still more accurate.

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[1]) In my paper of 1915 the equation for $\Delta \varepsilon$ was not derived independently, but adopted from DARWIN. There is however exactly at this point a numerical mistake in DARWIN's work, in consequence of which the value of ε derived from his formula is erroneous. Moreover DARWIN's computations are based on ROCHE's hypothesis, which gives an incorrect value for q. The formula (33) in the text must thus be used.

[2]) Neue Formeln für den Verlauf der Schwerkraft im Meeresniveau beim Festlande, Sitzungsberichte Berlin, 1915, p. 676.

6. The actual distribution of density within the earth is unknown. We can make hypotheses regarding that distribution, and by their aid compute the different integrals occurring in the theory of CLAIRAUT, and thus arrive at values of the different quantities, which can be determined by observation, such as ε, J, H and the surface density δ_1. The equation (5) is independent of the inner constitution. Consequently any hypothesis, which will reproduce any one of the two quantities ε and J will also give the correct value for the other. There are thus three conditions to be satisfied by any hypothesis on the distribution of mass.

ROCHE's hypothesis

$$\delta = \delta_0 (1 - k\beta^2)$$

contains only two constants, and can thus not be expected to satisfy the conditions. This expectation is confirmed by DARWIN's computations, which are based on this hypothesis. If we interpolate in his table (*Scient. Papers*, III, p. 112) for the correct value $\delta_1 = {\cdot}495$, we find $\varepsilon^{-1} = 288{\cdot}1$ and $q = {\cdot}5132$, which are entirely outside the limits of possibility.

WIECHERT's hypothesis, according to which the earth consists of a core and a crust, each of constant density, separated by a surface of discontinuity at which the density changes abruptly, contains three parameters, viz. the two densities and the radius of the surface of discontinuity. It is thus theoretically possible by this hypothesis to satisfy the three conditions, but it remains to be seen whether the values of the parameters, by which this is effected, are otherwise acceptable.

In order to test this and other hypotheses we must, as has been already said, compute the different integrals occurring in the theory of CLAIRAUT. As this theory is only applicable below the isostatic surface, we must take the radius of this surface as upper limit of the integrals. For the layers above it we can however replace the actual distribution of mass by an ideal one according to the theory of CLAIRAUT. The parts contributed to the integrals by the masses above the isostatic surface are then the integrals from S_0 to S_1. These have been computed taking for the density between S_1 and S_b: $\delta' = .186$ and below S_b: $\delta_1 = .495$, increasing regularly to δ_s at S_0. For the mean radii of S_0 and S_b, which are called s and b, I take $s = .98200$, $b = .99962$. The three conditions can then be enounced as follows

$$D_s = \frac{3}{s^3} \int_0^s \delta\beta^2 \, d\beta = 1{\cdot}04214 - {\cdot}02724\, \delta_s, \quad . \quad . \quad . \quad (35)$$

$$\frac{1}{s^5\,\varepsilon''_s}\int_0^s \delta \frac{d}{d\beta}(\beta^5\,\varepsilon'')\,d\beta = \cdot 87189 - \cdot 05008\,\delta_s\,, \quad . \quad . \qquad (36)$$

where we have put

$$\varepsilon'' = \varepsilon' + \frac{2}{7}\,\varepsilon^2$$

and for the isostatic surface we take the value corresponding to the adopted value of ε[1]), viz.:

$$\varepsilon''_s = {}^1/_{299\cdot 62},$$

finally

$$5\int_0^s D\beta^4\,d\beta = 1\cdot 16142 + \cdot 00077\,\delta_s - 1\cdot 250\,\frac{\lambda_1}{1+\lambda_1}. \quad . \quad . \quad . \qquad (37)$$

As an example I have applied these formulas to the theory of WIECHERT. The density above the surface of discontinuity thus is δ_s and below it $\delta_0 = \delta_s + \Delta$. The radius is taken $\beta_2 = ps$. The first two conditions give

$$p^3\,\Delta = 1\cdot 04214 - 1\cdot 02724\,\delta_s\,, \quad . \quad . \quad . \quad . \quad (38)$$

$$p^5\,\Delta\,(1-\xi) = \cdot 87189 - 1\cdot 05008\,\delta_s\,, \quad . \quad . \quad . \quad (39)$$

where we have put

$$\varepsilon''_2 = \varepsilon''_s\,(1-\xi).$$

For the determination of ξ I use the equation (10) for the surface of discontinuity $\beta_2 = ps$. This gives, using (38) and (39),

$$\Delta\,(1-\xi) - \left[\frac{5}{2}\delta_s - \cdot 00162\right]\xi + \cdot 0024\,\delta_s\,\xi^2 = 1\cdot 29750 - \cdot 99296\,\delta_s\,, \quad (41)$$

From the equations (38), (39) and (41) we can determine p, Δ, ξ, if δ_s is assumed. The computations have been carried out with two values of δ_s and the result is

for $\delta_s = 0\cdot 5$: $p = \cdot 8325$, $\Delta = \cdot 9160$, $\xi = \cdot 0531$;

$0\ 6$: $\cdot 7841$, $\cdot 8831$, $\cdot 0762$.

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[1]) The computation was first carried out for the approximate value $\varepsilon''_s = {}^1/_{300}$. This led to:

for $\delta_s = 0.5$: $\lambda_1 = + -.0007$,

0.6: $-.0006$.

The computation was then repeated with the exact value of ε''_s. In the original Dutch communication only the first approximate computation was included in the text, and the final one mentioned in a footnote. For more details regarding these computations the reader is referred to *B. A. N.* 55.

Finally we have

$$5\int_0^s D\beta^4\,d\beta = s^5\left[\delta_s + \frac{5}{2}\Delta p^2 - \frac{3}{2}\Delta p^5\right],$$

from which we find by (37)

for $\delta_s = 0 \cdot 5: \quad \dfrac{\lambda_1}{1+\lambda_1} = +.0003,$

$0 \cdot 6: \quad +.0004.$

Both these values agree with the adopted value $\lambda_1 = +.00044$ within the limits of the uncertainty of the data on which the computation is based. We must thus conclude that WIECHERT's hypothesis even in its simplest form, with only one surface of discontinuity and constant densities below it and between it and the surface of discontinuity, represents a possible constitution of the earth.

Mathematics. — „*Ueber die Zulassung unendlicher Werte für den Funktionsbegriff.*" By Prof. L. E. J. BROUWER.

(Communicated at the meeting of January 26, 1924).

Unter einer *endlichen Funktion* versteht man ein Gesetz, das jeder reellen Zahl x *einer gewissen Spezies* eine reelle Zahl $y=f(x)$ zuordnet; unter einer *unitär beschränkten Funktion* eine endliche Funktion, für welche $-1 \leq y \leq 1$ ist; unter einer *reduziert unitären Funktion* ein Gesetz, das jeder reellen Zahl x einer gewissen Spezies einen Punkt $y=f(x)$ des Zirkularkontinuums zuordnet, das aus dem Kontinuum der reellen Zahlen y mittels Identifizierung von y und $y+1$ hervorgeht; unter einer *vollen Funktion* eine Funktion, bei welcher die Zuordnung für jede *beliebige* reelle Zahl x stattfindet.

Um nun auch elementaren Relationen wie $y=\frac{1}{x}$ den Charakter einer vollen Funktion erteilen zu können, wird man zu den Begriffen der *infinitären Funktion* und der *reduziert infinitären Funktion* geführt, bei denen für $f(x)$ auch unendliche Werte zugelassen sind.

Diese Begriffe können aber nicht einfach definiert werden als Gesetze, die jeder reellen Zahl x einer gewissen Spezies ein $y=f(x)$ zuordnen, das im ersteren Falle entweder den Wert $+\infty$, oder den Wert $-\infty$, oder eine reelle Zahl, im letzteren Falle entweder den Wert ∞, oder eine reelle Zahl darstellt. So nämlich würde die Relation $y=\frac{1}{x}$ noch immer keine volle (reduziert infinitäre) Funktion liefern, weil sie der reellen Zahl r (vgl. Journal f. d. reine u. angewandte Mathematik 154, S. 3) weder den Wert ∞, noch eine reelle Zahl zuordnet.

Um dieser Schwierigkeit zu entgehen, muss man seine Zuflucht nehmen zu einer Definition folgender Art (die auch in Bd. XIII N°. 2 der Verhandelingen dieser Akademie, Eerste Sectie, im letzten Absatz von S. 17 zu lesen ist):

Unter einer *infinitären* bzw. *reduziert infinitären Funktion* wird das Resultat verstanden, das herauskommt, wenn eine unitär beschränkte bzw. reduziert unitäre Funktion der „infinitär ausdehnenden Transformation"

$$x'=x; \quad |y'|=-\log\{1-|y|\}; \quad yy'\geq 0$$

unterzogen wird.

Chemistry. — *"On the viscosity of rubber in ordinary and in acidified benzene"*. By O. de Vries. (Communicated by Prof. P. van Romburgh).

(Communicated at the meeting of May 3, 1924).

Viscosity determinations regularly made in our investigations on the preparation of rubber led us to formulate the following three groups of factors as having an influence on the viscosity of 1 °/₀ solutions of rubber in benzene (benzol)[1]:

1. By-substances of the rubber such as traces of coagulants or other chemicals, serum substances or their decomposition-products etc. These substances generally are also positive or negative catalysts of vulcanisation; they have an influence, therefore, on the rate of cure and the viscosity of the rubber, but leave the mechanical properties after vulcanisation (tensile strength and slope of the stress strain curve) unchanged.

2. Physical or mechanical changes in the colloidal rubber mass, by which the chemical composition of the rubber remains unchanged, but the consistency changes. These are accompanied by changes in swelling properties and in viscosity, but the rate of cure and the mechanical properties after vulcanisation remain unchanged. Into this group have to be brought the changes caused by heating (oxidation being excluded), plasticising and similar operations.

3. Changes in the rubber-hydrocarbon. These may be of two types, viz:

a. the rubber contains a hydrocarbon of other composition;

b. the rubber contains the normal hydrocarbon but it is decomposed (depolymerised, oxidised etc.).

The first case might perhaps furnish the explanation of the deviating properties of rubber from young trees, though proofs have not yet been produced. The second case includes different forms of tackiness and decay, by which the quality of the rubber deteriorates very much. These factors not only cause a marked decrease in viscosity, but also in the mechanical properties after vulcanisation; the rate of cure is also sometimes affected.

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[1] Archief voor de Rubbercultuur 2 (1918), 474; „Estate Rubber, its Preparation, Properties and Testing" (1920), page 592.

The differences in viscosity, which may be observed in different types of rubber, can nearly always be explained by one or more of the above groups of factors; it is, of course, possible that there are still other causes of changes in viscosity, not enumerated above.

To demonstrate that traces of different chemicals, as meant in group 1, may indeed influence the viscosity in benzene solution, we made in the beginning of 1918 some preliminary experiments by adding to 25 cc. of a 1 °/₀ rubber solution in benzene one or two drops of acetic acid, hydrochloric acid, ammonia, formalin or watery solution of different salts. It was found [1]) that indeed the viscosity increased or decreased just as when the same substances were added to the latex or to the rubber, and that the traces of these chemicals, taken up by the benzene-solution, influenced the viscosity to a marked degree.

These findings caused W. A. N. Eggink [2]) to make an extensive series of experiments on the influence of different electrolytes on the viscosity of rubber in benzene. Eggink showed that benzoic acid, acetic acid, sulphurous acid, mercuric chloride, hydrochloric acid and hydrogen sulphide decrease the viscosity, while ammonia first gives an increase and, in greater concentration, a small decrease; he determined the curves for different concentrations of these electrolytes. In his conclusions Eggink puts the question whether a determination of the viscosity of rubber in acidified benzene might not be of importance in the testing of rubber, because the rubber in such a solution is discharged and only shows the viscosity caused by its solvation, while the increase in viscosity caused by the electric charge is taken away.

An investigation in this direction seemed to us to be very promising; we investigated in the first place the way in which the factors which belong to the above meant three groups influence the viscosity in acidified benzene. It was found that the influence of chemicals and other by-substances (factors from group 1) may indeed be distinguished in a striking manner from the factors belonging to groups 2 and 3.

1. *Methodical.*

Eggink used for his investigations one sample of pale crepe, and dissolved about 1 gram in about 500 cc. benzene (the exact rubber content of the solutions was evidently not determined; it differed somewhat in different experiments, as may be seen from the differences in viscosity, but will have remained in the neighbourhood of $^1/_2\ ^0/_0$). The tests were made at 25° C. For obvious reasons we preferred to use solution of 1 $^0/_0$. tested at 30° C. after the prescriptions for viscosity determinations in use at the Rubber Experimental Stations [1]).

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[1]) "Estate Rubber", page 573.

[2]) Rec. Trav. Chim. Pays Bas 62 (1923), 317.

Hydrochloric acid was found, in EGGINK's investigations, to have the strongest discharging influence; at a concentration of about $1^1/_2$—2 millimol per liter the viscosity had decreased to a nearly constant value. We commenced our investigations therefore with benzol-hydrochloric acid and investigated how the viscosity changes with larger doses of hydrochloric acid than used by EGGINK. It was found that at higher concentrations (we generally used concentrations up to 75 millimol hydrochloric acid per liter) the viscosity remained also constant; at a concentration of about 5—10 millimol of hydrochloric acid per liter one is safely in the discharged zone, and relatively large differences in concentration of hydrochloric acid have therefore no influence on the viscosity. We therefore chose for our determinations a content of hydrochloric acid in that region and added, for instance, to a solution of 0.4 grms. of rubber in 45 cc. benzene, after filtering through cotton wool, 1.5 cc. of a 0.15 Norm. solution of hydrochloric acid in benzene (strength determined by titration in a large quantity of water), or about 5 millimol per liter.

The use of benzene-hydrochloric acid has two disadvantages. First the hydrochloric acid evaporates rapidly and the solution decreases in strength even from the repeated opening and closing of the bottle for taking out a certain portion. It is therefore necessary to check the strength from time to time; when one takes care to keep the hydrochloric acid content sufficiently high, this change in the dose of hydrochloric acid is of no practical importance because the viscosity at different contents of hydrochloric acid is constant.

A second drawback is that hydrochloric acid has a markedly deteriorating influence on the rubber. Traces of this acid, which remain in the dry rubber after coagulation cause it to become tacky in a short time; EGGINK already pointed out that the benzene solution of rubber even after a short time showed a decrease in viscosity. It was found that in the circumstances under which we work decrease in viscosity is about 0.15 in five minutes. By making the determinations always in the same manner the error can be kept practically constant (the time between the addition of the hydrochloric acid and the first reading of the time of flow is about 3 to 4 minutes); we have not applied a correction for this decrease in viscosity in the preliminary experiments to be described in this paper.

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[1]) Archief voor de Rubbercultuur 4 (1920), 124; Ind. Rubb. Journ. 59 (1923), 1197.

Especially the latter point is a drawback of this method of determination. We therefore investigated whether with other acids better results might be obtained. Acetic acid easily dissolves in benzene, but was found to discharge the rubber only at a much larger concentration (about 0.4 normal) and to have not so sure an effect as hydrochloric acid.

We hope to continue the experiments with other acids, amongst others formic acid, but have restricted ourselves in this first investigation to benzol-hydrochloric acid.

It is interesting to draw attention to the fact that the highest concentrations of acid (0.075 Norm. HCl and 0.405 Norm. acetic acid) do not cause the slightest trace of flocculation or coagulation in the benzol solution. In a watery emulsion (latex) of 1 °/₀ rubber content coagulation takes place at an acidity of 0.002 Norm. both for hydrochloric and acetic acid [1]. The difference caused by the small degree of solvation in water and the large degree of solvation in benzene is very apparent, bearing out the explanation given by Kruyt and van Bungenberg de Jong [2].

2. *Factors from group* 1.

As a first experiment we prepared, from portions of 100 cc. taken from one lot of well mixed latex, a series of samples which, according to the conclusions we had formulated, might be expected to show differences in viscosity due to factors from group 1 (coagulants, other chemicals, serum substances and their decomposition products). These experiments gave the results mentioned in table I; the amounts mentioned hold for 100 cc. of latex (undiluted or diluted).

The "ordinary" viscosity shows the differences that were to be expected. The normally prepared control sample (4656 B) was coagulated by 7 cc. of a $2^1/_2$ °/₀ acetic acid solution and creped (as all other samples with the exception of 4656 E and F) about 2 hours after coagulation in the ordinary way. This sample shows a normal figure for viscosity of about 30.

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[1] Proc. Royal Acad. Amsterdam, Vol. XXVI, p. 675.

[2] Zeitschr. phys. Chem. 100 (1922), 250.

TABLE I.

Sample	Description	Viscosity	
		ordinary	acidified
4656 L	5 grm sodium acetate, 14 cc. $7^{1}/_{2}$ % acetic acid	36.3	16.3
„ F	coagulated spontaneously creped after 1 week	34.5	15.9
„ E	„ „ „ „ 1 day	32.7	16.3
„ B	control, 7 cc. $2^{1}/_{2}$ % acetic acid	30.3	15.8
„ P	as B, after crepeing 2 hours in water of 90°	23.9	15.8
„ A	latex diluted 1:1, with 4 cc. $2^{1}/_{2}$ % acetic acid	29.0	15.9
„ G	21 cc. $2^{1}/_{2}$ % acetic acid	29.0	15.6
„ H	1 grm of alum	21.8	15.7
„ K	latex diluted 1:1, 0.5 cc. 3.4 Norm. sulphuric acid	16.9	15.7

The highest viscosity was found in sample 4656 L from latex to which a large dose of sodium acetate had been added, and which of course needed a large dose of acetic acid for coagulation. Sodium acetate, as all alkaline substances, causes an increase in viscosity; this increase is not nullified by the effect of the double quantity of acetic acid. This, therefore, is an example of the addition of a chemical that gives an increase in viscosity.

The next two samples (4656 E and F) were coagulated spontaneously (without acetic acid) and left 1 and 7 days respectively before being rolled. A decomposition of serum substances then takes place, which gives an increase in viscosity. Compared with the control sample the above three samples show the normal differences in "ordinary" viscosity, but in acidified benzene the figures are practically equal.

When the freshly rolled crepe is kept for two hours in hot water, the viscosity decreases considerably (sample 4656 P). Up till now one was in doubt whether this was caused only by an extraction of serum substances (factor from group 1), or whether a change in the colloidal state of the rubber caused by the temperature of 90° (factor from group 1) had also been at work. The viscosity in acidified benzene clearly shows that the rubber itself is unchanged and that in this case there has only been an extraction of serum substances.

Another example of a decrease in the amount of serum substances is provided by sample 4656 A: the latex was diluted with an equal volume of water and coagulated with 4 cc. of $2^1/_2\,^0/_0$ acetic acid per 100 cc. of diluted latex. The "ordinary" viscosity decreases somewhat, as usual, but the viscosity in benzene-hydrochloric acid remains unchanged.

The last three samples show the influence of acids and acid chemicals. The use of three times the ordinary amount of acetic acid causes a small decrease in viscosity; an ample dose of alum, used as a coagulant, gives the usual low viscosity, whilst a very large dose of sulphuric acid (about ten times the minimum dose necessary for coagulation) gives a very low viscosity. Nevertheless the viscosity in acidified benzene is, in these three cases, unchanged!

Reviewing the figures in table I, the contrast between the largely decreasing figures in column I and the nearly equal figures in column II is very striking. Especially in the case of alum the result is interesting, because some investigators have supposed coagulation with alum to give an inferior rubber. It will be clear from table I that this is not the case; the rubber itself is unchanged.

We prepared a second series of samples to test further the influence of the content of serum substances, by strongly diluting the latex. The results are contained in the following table.

TABLE II.

Sample	Dilution	$2^1/_2\,^0/_0$ acetic acid cc.	Rolled day	Viscosity	
				ordinary	acidified
4801 A	undiluted	7	same	41.9	19.5
„ B	„	7	next	42.7	20.3
„ C	„	4	„	43.7	20.3
„ D	1 : 1	4	„	40.3	20.3
„ E	1 : 4	2	„	38.5	19.2
„ F	1 : 9	1.2	„	35.0	19.5
„ G	1 : 19	0.6	„	35.5	19.0

The first sample is the control, prepared in the same way as in series 4656 but from another latex. Sample 4801 B was coagulated by the same amount of acetic acid, but rolled the next day; this gives a small increase in viscosity. Sample 4801 C was also rolled the next day but with a smaller amount of acetic acid (a sufficient dose when rolling 24 hours after coagulation) and therefore gives a somewhat larger viscosity. The other samples show clearly a decrease in viscosity as the dilution of the latex increases and the rubber therefore retains less serum substances during coagulation.

The last column again shows very clearly that the viscosity in acidified benzene is practically the same for all samples. Whether the small differences from 20.3 to 19.0 have a certain significance or whether they must be ascribed to errors of determination can only be determined by a more extensive investigation.

The influence of traces of acid used for coagulation is very clearly seen from the following example.

TABLE III.

Sample	Coagulant	Viscosity	
		ordinary	acidified
4862 A	acetic acid, 5 cc.	32.3	16.6
„ B	„ 10 „	31.4	16.1
„ C	„ 20 „	29.8	16.0
„ D	formic acid, $2^{1}/_{2}$ „	31.1	16.6
„ E	„ 5 „	30.0	16.5
„ F	„ 10 „	27·3	16.5
„ G	oxalic acid, 4 „	30.8	16.6
„ H	„ 8 „	24.5	16.6
„ I	„ 12 „	19.9	15.3

The latex was diluted to 20 $^{\circ}/_{0}$ rubber content and coagulated by the above mentioned amount of $2^{1}/_{2}$ $^{\circ}/_{0}$ acid per 100 cc. diluted latex.

An increase in the amount of acetic acid gives the usual small decrease in "ordinary" viscosity. Formic acid is a stronger coagulant and gives, in larger doses, a larger decrease in "ordinary" viscosity. The non-volatile oxalic acid gives a considerable decrease in "ordinary" viscosity. The figures for „acidified" viscosity are, however, the same for all samples, showing that the differences in "ordinary" viscosity are only caused by the traces of acid absorbed by the rubber.

3. *Factors from group* 2.

That factors from group 1 give a charge to the rubber can therefore be demonstrated very clearly by a comparison between the "ordinary" and the "acidified" viscosity. In order to investigate how the figures are affected when factors from group 2 play a role we made the following experiments.

Heating. Heating, provided no oxidation takes place, causes a disaggregation and weakening of the rubber without the quality really deteriorating and especially without the properties after vulcanisation suffering from it. Viscosity and swelling properties, however, change very much, and it was therefore to be expected that the viscosity in acidified benzene would decrease. This was indeed found to be the case.

TABLE IV.

Sample	Description	Viscosity	
		ordinary	acidified
4656 B	Control	30.3	15.8
„ P	fresh crepe 2 hours in water of 90°	23.9	15.8
„ M	fresh crepe dried in vacuo at about 80°	21.6	14.3
4801 A	Control	41.9	19.5
„ I	fresh crepe dried 2 hours at 90°	35.0	19.5
„ Q	dry crepe 2 hours at 90°	28.4	17.3

Soaking of the freshly rolled crepe in water at 90° (sample 4656 P) has already been discussed above; the viscosity decreases but the rubber itself remains unchanged, as may be seen from the viscosity in acidified benzene. Heating the freshly rolled, wet crepe for two hours in hot air at 90° causes a decrease in ordinary viscosity, but the viscosity in acidified benzene remains unchanged (sample 4801 I). The still moist crepe is, in agreement with other observations, relatively insensitive to heat.

When heating air dry crepe however, whether the fresh crepe is heated for some time in vacuo (sample 4656 M) till it is quite dry but also rather weak, or whether the air dry crepe is heated in a drying oven at 90° C. (sample 4801 Q), the viscosity in acidified benzene also decreases and the colloidal state of the rubber therefore changes.

There is no certainty, however, that in these causes one has exclusively to do with a change in the colloidal properties, because a slight degree of decomposition of serum substances is not excluded during drying; the colour of the rubber may darken markedly and the smell changes. Oxidation could only be excluded with certainty by repeating the experiment in a closed tube filled with an inert gas. The results obtained in these experiments would therefore have to be controlled. The fact that the properties after vulcanisation remain unchanged makes it probable that the rubber itself is not affected.

Plasticising. A second example of a factor belonging to group 2 and causing a decrease in viscosity without affecting the properties after vulcanisation, is plasticising or mastication. Its influence may be seen from the following experiment:

TABLE V.

Sample	Description	Viscosity	
		ordinary	acidified
4801 B	Control	42.7	20.3
„ M	same, creped twice as long	39.0	18.9
„ N	„ , „ four times as long	24.0	14.0
4801 F	Control (latex 1 : 9 dilution)	35.0	19.5
„ O	same, creped twice as long	30.0	18.4
„ P	„ , „ four times as long	17.5	13.0
4656 B	Control	30.3	15.8
„ C	same, creped very long (10 minutes)	8.2	6.3
„ Q	dry crepe slightly plasticised (5 times)	22.2	13.1
„ R	„ „ plasticised longer (10 times)	2.7	2.2

When continuing the crepeing and washing of the coagulum on the rolls longer than usual the viscosity decreases. As the cause one might consider a further removal of serum substances; but as the rate of cure does not decrease or only to a small degree by this prolonged rolling, we have in former years sought the explanation in a beginning of plasticising. The decrease of viscosity in acidified benzene shows that this explanation was correct. When crepeing for a very long time, so that the crepe becomes markedly sticky (sample 4656 C) the viscosity in acidified benzene even decreases very much.

When plasticising the dry crepe on hot rolls, as is done before mixing in the sulphur, the first five times of rolling, when rolls are still rather wide apart (sample 4656 Q), cause a distinct decrease in both viscosities; prolonged plasticising (sample 4656 R) causes both the ordinary and the acidified viscosity to decrease to a very low value.

4. *Factors from group* 3.

Different factors may affect the rubber so strongly that a real decay sets in and that the properties after vulcanisation also suffer, or even that a proper vulcanisate can no longer be obtained. One may, of course, expect that these factors would also cause a decided decrease in viscosity in acidified benzene.

As a first example we investigated the influence of sunlight by hanging a crepe prepared on September 26 for about three hours in the sun on September 27, 28 and 29, that is during the time the crepe was drying. The following figures were obtained:

TABLE VI.

Sample	Description	Tested	Viscosity	
			ordinary	acidified
4656 B	Control	—	30.3	15.8
4656 N	In the sun	8 October	10.3	6.1
	same, kept in the dark	26 „	9.3	6.0
	„ „ „ „ „ „	10 December	9.2	6.4
	„ „ „ „ „ „	10 January	8.6	6.3

The figures decrease very much by sunning; when keeping the rubber afterwards in a closed box the ordinary viscosity decreases still a little further, but the viscosity in acidified benzene remains constant.

A second example of deterioration is provided by the tackiness caused by oxidation when traces of a copper salt are present. The rubber at first becomes sticky and then gradually resinifies. We found the following figures:

TABLE VII.

Sample	Description	Tested	Viscosity	
			ordinary	acidified
4656 B	Control	—	30.3	15.8
4656 O	as B, 0.1 grm of $CuSO_4$ 5 aq added to the latex	8 October	15.6	9.3
		27 „	9.0	6.7
		16 November	5.8	5.0
		6 December	3.4	3.5
		5 January	2.3	2.2
		5 February	1.6	1.5

Therefore not only a marked decrease during the time the crepe hung to dry (26 September—8 October) but also a further decrease when keeping the crepe between sheets of paper in a wooden box or cupboard, till at last the viscosity in ordinary benzene is equal to that in acidified benzene and one has no longer to do with rubber but with different resinous oxidation products.

Traces of hydrochloric acid also attack the rubber very severely and lead to tackiness and resinification. The viscosities were;

TABLE VIII.

Sample	Description	Tested	Viscosity	
			ordinary	acidified
4656 A	Control, prepared 26 September	—	29.0	15.6
4656 I	latex 1 : 1, 17½ cc. HCl 3.1 norm.	6 October	14.0	13.0
		19 „	13.3	11.9
		6 December	9.6	9.3
		5 January	7.6	7.6
		5 February	6.0	5.5

Both crepes were dried in the air in subdued daylight. The very large dose of hydrochloric acid used for coagulation (about 0.46 Normal on the latex, i.e. well into the second zone of coagulation) causes a decrease in ordinary viscosity to nearly the value of the acidified viscosity; in other words the traces of hydrochloric acid retained by the dry rubber were sufficient to discharge it nearly completely in benzene. During the time the rubber hung to dry the viscosity in acidified benzene decreased from 15.6 to 13.3, and by further keeping in the dark it continued to decrease.

Rubber that contains traces of hydrochloric acid gets tacky in the light much quicker than in the dark. We collected the following figures:

TABLE IX.

Sample	Description	Tested	In dark		In light	
			ordinary	acidified	ordinary	acidified
4801 A	Control, prepared 14 Nov.	—	41.9	19.5	—	—
4801 R	Coagulated with 35 cc. HCl 3.1 Norm.	28 Nov.	18.4	18.0	—	—
		5 Dec.	18.1	18.3	14.8	14.6
		12 „	18.8	18.2	11.9	12.2
		19 „	18.0	19.0	12.1	11.6
		12 Jan.	17.2	17.3	6.4	5.5
		12 Febr.	17.2	15.9	3.0	2.0
4801 D	Control, latex, 1 : 1, prepared 14 November	—	40.3	20.3	—	—
4301 H	Latex 1 : 1, 17½ cc. HCl 3.1 Norm.	28 Nov.	17.7	17.1	—	—
		5 Dec.	18.1	16.5	13.9	14.7
		12 „	17.6	17.0	12.3	12.8
		19 „	18.8	16.6	11.4	9.9
		12 Jan.	17.2	16.4	4.5	3.0
		14 Febr.	17.0	18.2	4.0	3.6

The control samples were dried after coagulation in ordinary, subdued daylight, but the crepes coagulated by hydrochloric acid were hung to dry in a dark room. The viscosity, during the time of drying, decreased much less than in the former experiment (table VIII), but both viscosities had already decreased somewhat when the rubber was tested for the first time on November 28. From that date onwards half of the samples was enveloped in a piece of paper and kept in a well closed wooden box and only exposed to the light for a moment when sampling for the tests.

The other half was laid on a table, exposed to ordinary daylight and even sometimes, in the early morning, to direct sunlight. The figures in table IX each relate only to one determination, and are not averages of two or three determinations as is the case with the figures in all the other tables; the errors of determination play a clearly visible role and cause in one case the ordinary, in another case the acidified viscosity to show a slightly higher figure. The slow decrease in both viscosities when keeping in the dark, and rapid decrease when keeping in the light is nevertheless clearly seen from table IX.

Sample 4801 R, coagulated with the largest amount of hydrochloric acid, shows nearly the same value for both viscosities, indicating that the rubber in ordinary benzene was completely discharged by the traces of coagulant retained by it. Sample 4801 H shows nearly always a slightly higher figure for viscosity in ordinary benzene and was evidently not completely discharged.

5. *Other cases.*

Different factors in preparation have a complicated influence on the properties of the rubber. While in the cases discussed above an explanation could already be given on the basis of earlier experiment, and the factor causing a change in viscosity could be identified with a reasonable degree of certainty, so that the conclusions drawn in 1918 had only to be confirmed by the present investigation on the viscosity in acidified benzene, there are other cases in which an explanation could hitherto not be given. We tried to solve these points also by using the new method.

Coagulation by heating the latex, for instance, gives a rubber with a low viscosity, a large rate of cure and unchanged mechanical

TABLE X.

Sample	Description	Viscosity	
		ordinary	acidified
4656 B	Control	30.3	15.8
„ S	coagulated by heating	23.6	16.2
4801 A	Control	41.9	19.5
„ L	coagulated by heating	29.2	20.2

properties after vulcanisation. Generally a quicker rate of cure is accompanied by a larger viscosity; that the viscosity in this case shows a decrease might be explained by the effect of heating causing a decrease larger than the increase from the included vulcanisation-catalysts.

The decrease in "ordinary" viscosity is in both cases very marked, but the "acidified" viscosity shows no decrease and a factor from group 2 has therefore not been at work. The hypothesis put up in former years was therefore not the correct one, and it seems that we have to do, in this case, with some serum constituent that decreases viscosity but causes an increase in rate of cure.

Complicated are also the changes caused by coagulation with alcohol. One obtains a rubber with a higher ash content, a lower viscosity, a somewhat larger slope of the stress strain curve (that is a somewhat weaker vulcanisate), an unchanged tensile strenght and rate of cure. An explanation for this combination of properties could not yet be given. We found:

TABLE XI.

Sample	Description	Viscosity	
		ordinary	acidified
4656 B	Control	30.3	15.6
„ T	Coagulated with $^1/_2$ vol. alcohol	23.0	13.9
4801 A	Control	41.9	19.5
„ K	Coagulated with $^1/_2$ vol. alcohol	34.7	19.3
4979 A	Control	29.1	15.6
„ K	Coagulated with $^1/_2$ vol. alcohol	21.8	14.2
„ L	„ „ 1 vol. „	21.7	14.5

The "ordinary" viscosity has therefore decreased markedly in all three experiments by the coagulation with alcohol; the viscosity in acidified benzene decreased somewhat in two out of the three cases. The lower value for the "ordinary" viscosity is therefore only partly to be ascribed to a higher content of electrolytes, and is partly caused by a (not yet further explained) change in the rubber itself.

Rubber from matured coagulum (coagulum that is kept for some time in the wet state, resulting in a decomposition of serum substances by micro-organisms, had formation of vulcanisation accelerators) often shows a very high viscosity and a somewhat higher tensile strength than ordinary plantation rubber. Some investigators hold the opinion that the inner properties of such rubber are better; we have ascribed the higher tensile strength to the fact that such rubber vulcanises very quickly and therefore is exposed for a shorter time to the deteriorating influence of the high temperature of vulcanisation.

When determining the viscosity of such rubber in acidified benzene one finds normal figures. For instance:

TABLE XII.

Sample	Viscosity	
	ordinary	acidified
4329 AW	53	16.8
„ BW	75	17.6
4766 AW	65	21.7
„ BW	67½	19.4
„ CW	58	20.1
„ DW	53	19.3

Control samples of ordinary crepe, prepared from the same latex, were not available, but it may be clearly seen that the high values for ordinary viscosity completely disappear when acidifying the benzene; the figures for acidified viscosity are the same as found for ordinary samples of rubber, so that the high figures may be ascribed to a change caused by the (alkaline?) decomposition products of serum constituents (proteins?).

Rubber from young trees shows strongly deviating properties, namely a low tensile strength and viscosity, a high figure for slope and a quick rate of cure. That the quality of such rubber is inferior may be ascertained by pulling the sheets by hand: the rubber is brittle or "short", breaks more easily than ordinary plantation rubber.

The viscosity was found to be: (See table XIII)

The ordinary viscosity is therefore low, but the viscosity in acidified benzene is still considerably lower, so that from these figures also it is clear that in such cases the rubber itself is inferior.

TABLE XIII.

Sample	Viscosity	
	ordinary	acidified
3985 A	7.0	4.4
„ Bw	6.4	4.4
4965	17.7	10.1
5121 A—D	20.7	12.0
4959	24.6	13.1

We intend to continue these investigations extending them to a number of other cases, and to compare the results with other properties, such as plasticity, which (according to preliminary determinations) seems to show a certain parallellism with the viscosity in acidified benzene.

6. *Conclusion.*

From the foregoing it will be clear that a determination of the viscosity in acidified benzene, besides the viscosity in ordinary benzene, promises to become a valuable acquisition in the study of the factors that influence the properties of rubber.

Whether this determination may become of importance for the ordinary testing of rubber will have to be investigated further. In that case one very often has to do with first quality product, in which the properties of the rubber itself are mostly good, but in which the content of by-substances (traces of chemicals, serum constituents etc.) may vary according to the method of preparation and may give an undesirable variability in rate of cure. In such testing work the ordinary viscosity will give valuable indications, but the viscosity in acidified benzene will not show large differences.

When, however, one has to do with the lower grades of rubber (for instance the brown types), important information may be expected from the viscosity determined in acidified benzene, which will allow a more reliable conclusion as to the quality of the rubber than the viscosity in ordinary benzene.

When testing special types of rubber, prepared by other methods than those for the ordinary plantation product, the viscosity in acidified benzene will doubtless prove a very valuable asset.

Buitenzorg, February 1924. *Central Rubberstation.*

Palaeontology. — *"On the Principal Characters of the Cranium and the Brain, the Mandible and the Teeth of Pithecanthropus Erectus".* By Prof. Eug. Dubois.

(Summary of the communications at the meetings of May 26, and November 24, 1923, nearly identically summarized in the "Verslag" of the meeting of February 23, 1924)[1].

Before the morphological characters of the fossil remains of this Primate are discussed, a few remarks may be made about the state of perfect **mineralization** in which they are. This state is entirely different from that of the oldest of the human remains known. Their specific gravity, like that of the bones of the other mammals which were dug up with them at Trinil (which bones have the same outward appearance), has risen to about 2.7, an increase of about 35 °/₀ compared with dry fresh bony substance. That of the Piltdown-man (Eoanthropus) differs very little, if at all, from the latter, according to the accurate determination of its specific gravity[2]). Also the mandible of Mauer and the bones of the fossil man of La Chapelle-aux-Saints have become only comparatively little heavier. On the other hand the specific gravity of the bony substance of the fossil mammals from the late-pliocene clay of Tegelen (with Elephas meriodionalis, Rhinoceros etruscus, Equus stenonis, pliocene Deer, Trogontherium cuvieri, and a pliocene flora), though petrified in another way, viz. silicified, is equal to that of the bones of Trinil. In the fossil bones of the Neandertal man of Spy a not unimportant quantity of glutinous substance has remained; in the bones of Trinil, on the other hand, only traces of humus substances are present, which give them a chocolate-brown colour.

According to the analysis of the late Prof. J. M. van Bemmelen both phosphate and carbonate of calcium have taken the place of the ossein, and they contain fluorium in the quantity which, according to Ad. Carnot's investigations, is characteristic of fossil bones of the Pliocene. The particular pseudomorphism known as mineralization, petrifaction or fossilization, has very strongly affected the bones of Trinil. Their great antiquity appears also from this that calcite and pyrites have crystallized in many cavities and canals in the bones.

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[1]) The author proposes before long to discuss the subjects of these communications more at length in a memoir elucidated by illustrations.

[2]) Quarterly Journal of the Geological Society, Vol. 69, p. 121. London 1913.

Though these physical and chemical properties with the character of the mammal fauna, stamp the remains of Pithecanthropus as pliocene, this says little as criterion of its phylogenetic significance, given the simultaneous existence of allied forms at different stages of development, observed everywhere.

Decisive are here the **morphological characters** of Pithecanthropus. In the first place those characters of the skull which can be recognized at the calvarium from the Kendeng-layers of Trinil, and of the brain in so far as these can be judged from the endocranial cast of the calvarium; further the morphological characters of the mandible and the teeth, from a fragment found near Kedung Brubus, in the same Kendeng-layers, and the three teeth dug up at Trinil; and finally the morphological characters of the femur excavated there.

The morphological investigation of the cranium is restricted or hampered by three circumstances: first that only the upper part of the calvaria as calotte or calvarium has been preserved; secondly that the outer surface has been greatly corroded by sulphuric acid, formed from pyrites in the volcanic tufa; thirdly that the cranium has been deformed in a natural way (through trigonocephalism, though in a small degree).

These circumstances have not seldom led to erroneous conclusions in the study of the plaster cast. At the fossil calvarium itself at least the principal morphological characters can be clearly observed.

At the plaster cast it cannot be seen in every detail to what extension the calvaria has been preserved, especially not at the frontal border and in the temporal region. The situation of the glabella-point and of the asterion, and in approximation of missing parts, as the meatus acusticus, the processus mastoideus, can therefore only be determined at the fossil itself. In virtue of this and on account of some loss of substance at the inion, the real maximum cranial length may be put at 184 mm., though the directly measurable length is only 180.5 mm.

Particularly, however, about the original state of the external surface of the calvaria the fossil alone can give an accurate idea. It must have been smooth on the whole, about as at the skull of a small gibbon species, because there are nowhere traces present of distinct ridges, still less of crests, except the apparently perfectly gibbonlike crista supramastoidea. Between intact spots, corresponding to which also the endocranial cavity reaches it greatest width, the greatest cranial breadth, 131 mm., can be measured. Hence the length-breadth index is 71.2. It may also be mentioned here that the calvarial height above the glabella-inion line (SCHWALBE's "Calottenhöhe") is 61 mm.

On account of much loss of substance on the left side the post-orbital breadth (constriction) can also only be judged from the fossil remain itself. It is 87 mm., the original (true) breadth must, however, have been at least 91 mm. The fronto-biorbital index (SCHWALBE), i.e. its ratio to the external orbital facial breadth, which latter I estimate at 115 mm. as a minimum, is at most 79. In a cranium of a Hylobates agilis resembling that of Pithecanthropus in many respects, this index is 78.4. The post-orbital length index (SCHWALBE) is 25.5, the distance between the orbital constriction and the bregma-transversal being 47 mm. This index is 25 at the same cranium of Hylobates agilis. The whole pre-cerebral part of the frontal bone is hylobatoid, like the rest.

The keel-shaped elevation of the external surface of the frontal bone is about the same as that in the original state, but the rhombic eminence at its upper end, whose apex coincides with the bregma, was much less pronounced at the intact cranium than at the corroded calvarium. The trigonocephalism, which was the cause that the frontal part of the skull is comparatively narrow, the temporal part comparatively broad, is caused in Man by early fusion of the two frontalia; it may be assumed that also in the skull of Pithecanthropus this premature fusion has given rise to the existence of the torus frontalis medianus, in favour of which also plead the extraordinarily strong impressions of the cerebral convolutions at the inner surface of the frontal bone. Hence the great breadth of the temporal part of the skull is not to be considered, as ELLIOT SMITH asserts, as a consequence of spontaneous stronger development of the temporal cerebral lobe.

The shortest distance between the two temporal lines was probably 85 mm., the ratio to the cranial breadth is the same as that of the small gibbon species.

It is seen that to the sagittal arc-length of the cranial vault the frontal bone contributes 100 mm., the parietal bone 90 mm., and the upper part of the tabular portion of the occipital bone 45 mm. This is an entirely different ratio between the two first divisions of the vault from that in Homo sapiens and Homo neandertalensis, where the parietal arc is longer than, or equally long as, the frontal arc. In the Hylobatidae, on the other hand, the parietal bone is much shorter in comparison with the frontal arc, than in Pithecanthropus. The latter's fronto-parietal index is 90, that of the large gibbon genus, Symphalangus, 53 on an average, and that of the small gibbon species has an average value of 42. The relative length of the frontal squama diminishes with increasing size of the body, in consequence of the diminution that this entails of the ratio between the volumes of the orbita (with the eye) and the cranial cavity (with the brain). Besides, in Pithecanthropus the cranial cavity has become more spacious mostly in another way than through the greater size of the body.

The lower part of the tabular portion of the occipital bone, the pars nuchalis, bends downwards and forwards at an apparently not very obtuse angle. But this obtuse angle was much larger at the intact skull, because at the fossil calvarium the loss of substance greatly increases towards the edge of the fragment, so that this edge only still consists in the knife-like lamina interna. In the intact state of the skull Pithecanthropus resembled the Hylobatidae in the steepness of the planum nuchale.

The torus occipitalis presents the closest resemblance with that of Symphalangus syndactylus.

In the view of the cranial vault from below the very spacious and largely communicating right and left sinus frontales may be recognised in their full width and depth. Together they have a width of 55 mm.; their greatest depth (measured from the front backwards) is 23 mm. The maximum endocranial length between the frontal and the occipital poles amounts to 153 mm. on the left, to 155 mm. on the right side. The maximum endocranial breadth is 124 mm. The apex of the endocranial cavity is above this transversalis, to a height of 58 mm. This transversalis lies about in the transversal plane of the right frontal pole and the upper edge of the right sulcus transversus, to which plane the capacity of the calvarium was measured. Within the whole reach of the frontalis its inner surface shows very strong impressions of the cerebral convolutions, undoubtedly a consequence of premature union of the two halves of the bone. The cerebral impressions and the likewise very strong grooves of the arteria meningea medea are most clearly to be seen at the endocranial cast, reproducing the positive of the cerebral surface. In the reach of the parietalia and of the attached parts of the temporalia there are hardly traces to be seen of the impressiones digitatae and juga cerebralia, with the exception of the jugum sulci centralis and the jugum sulci intraparietalis. There is also a strong jugum sulci lunati, lying immediately behind the sutura lambdoidea. (In the Hylobatidae and the Chimpanzee this jugum lies as a rule immediately before the suture).

Of that process in the cranial cavity chiefly formed by the lesser wing of the sphenoid, which in the brain corresponds to the fissura Sylvii, a considerable piece has been preserved especially on the right side. Present is also a powerful crista occipitalis interna, which possession distinguishes Pithecanthropus from the Hylobatidae, which have there a wide groove, as impression of their round, barer vermis cerebelli, and agrees with the large Simiidae and Man. The endinion lies 26 mm. lower than the inion externum. The sulcus sagittalis, the torcular Herophili, and the right sulcus transversus are wide and deep. With respect to the latter sulcus the parieto-mastoid suture lies exactly the same as in Hylobates. Also as regards the situation of the internal asterion Pithecanthropus agrees entirely with Hylobates. For the postasterial index (the ratio of the distance between the asterion and the occipital pole and the endocranial length) I find 15,5 in Pithecanthropus, 15,8 in Hylobates, and a mean value of 24 in human skulls of different races.

The form of the skull of Pithecanthropus is on the whole not human; nor is it a transition of any type of manlike apes to the human type. The agreement with the anthropoid cranial type, particularly that of the small gibbon species, of the genus Hylobates, may on the other hand be called perfect, taking into consideration the inevitable deviation in the proportions in consequence of the ratio of the volume of the brain and the eye varying with the increasing bulk and cephalisation. For with increasing bulk the eye increases somewhat less in volume than the brain, and by the much higher cephalisation of Pithecanthropus the brain was besides enlarged far beyond the homoneuric ratio. The fossil cranium is not more highly arched, has no less receding forehead, and the pre-cerebral part of the frontalis projects equally far forward as in those Apes. The constriction ("Einschnürung") behind the orbitae is also perfectly pithecoid in its depth and its situation at a greater distance from the front border of the skull; so is the place where the external auditory meatus must have been, and the form of the crista supramastoidea. Perfectly pithecoid was further the shape of the torus occipitalis transversus and the value of the angle at which the nuchal plate of the occipital bone bends foreward and downward. In all these points Pithecanthropus is distinguished no less strongly than the Anthropoid Apes from the Neandertal Man. From the latter character of the fossil skull it may be derived that also the condyles of the occipital bone were placed in the same way as at the skull, so that the head was not equipoised on the spinal column as in modern Man, but was carried by strong nuchal muscles and ligaments as in Apes. It is not to be seen by the structure of the skull that Pithecanthropus deserves the name of *erectus,* assigned to him on account of the features of the femur. This is, however, to be derived from a character of his brain and another of his mandible still to be discussed.

It is certain that the erect posture of the body, which clearly appears from the shape of the femur, was not such a perfect one as in Man; the correlation, at least, did not extend to the skull.

Nor can the skull, however, have belonged to an Anthropoid Ape, because the relatively very large skull as regards shape presents a close, nay striking resemblance with the skull of a small Hylobates-species, the smallest of the Anthropoid Apes, whereas judging not only from the femur and the molar teeth, but also from the skull itself, Pithecanthropus must have surpassed the size of a large chimpanzee, and very much that of a middle-sized man. Those smallest Manlike Apes distinguish themselves especially by their large neurocranium in the proportion of their splanchnocranium, the facial part of the skull. This is a consequence of the law governing the relation between the quantity of the brain and the bulk of the body in closely allied species. Small species have in general larger brains in comparison with their body weight than large ones of the same genus, sometimes also of the same family, in general than large homoneuric species (species with the same organisation of the nervous system). Judging by the linear dimensions, and as will appear further, by the cranial capacity, Pithecanthropus *as an Anthropoid Ape* would have been a giant of about 300 kg. weight, much larger than the heaviest gorilla. But Pithecanthropus was *not* such a giant. This appears not only from the dimensions of the femur, but also in the skull from the great distance of the temporal line, the boundary of the surface origin of the musculus temporalis from the median line, an indication that this masticatory muscle was weak with respect to the size of the neurocranium, though in such a gigantic Anthropoid Ape as Pithecanthropus then must have been, it would have had on the contrary a comparatively much larger area of origin, to find sufficient place at the then *relatively* small neurocranium. We may refer to the cranial crests of large male gorillas and orang-utans.

That the fossil skull bears such a striking resemblance to that of Hylobates, this dwarfish genus among the Anthropoid Apes, does not, therefore, compel us to class Pithecanthropus for this reason among this family, but it also gives support to the view that the Hylobatidae are actually to be considered as genuine and then the most primitive Manlike Apes, though such as are particularly specialized by their long arms and sabre-shaped canine teeth.

The most important difference between the brain of the Hominidae and that of the Simiidae, the Anthropoid Apes, consisting in the difference of volume, it is very desirable to know the **cranial capacity** of Pithecanthropus. The volume of the space in which the greater part of the cerebral hemispheres was contained, can be accurately measured with water up to a certain morphological limit [1]) in the calvarium, this having been made watertight. A volume was found of 570 cm^3. In order to calculate the whole capacity from this part, the ratio of morphologically the same part to the whole capacity was determined in skulls of apes which resemble the skull of Pithecanthropus as much as possible. In general this ratio lies in skulls of apes very near 1 : 1.6. In human skulls of different races 1 : 1.4 was found on an average. In the skull of a Hylobates agilis, the shape and structure of which closely resembles the fossil skull, this ratio is 1 : 1.56. Taking small morphological differences into account, as particularly the much greater impressions which the orbitae make in that upper cranial part in Hylobates, the cranial capacity of Pithecanthropus can be calculated in approximation from this ratio at 900 cm^3. This calculated capacity can certainly not depart greatly from reality. Besides, it may be considered to be about the mean of the species, as will appear further below. It may be assumed that with equal body weight Pithecanthropus possessed double the brain quantity of the Anthropoid Apes.

In its side view (norma lateralis) the **endocranial cast** presents a striking resemblance with the endocranial cast of a small Hylobates-species reproduced at the same size; thus also in the steep position of the endocranial planum nuchale. The only difference of importance consists in the much smaller impression which the orbitae make in the cerebral hemispheres than in Hylobates (where there is present a very pronounced "bec encéphalique" = rostrum orbitale, ethmoidale, or cerebrale), as a consequence partly (about $^1/_4$ of the calculated surface dimension) of the circumstance discussed before, that with increasing body-weight the eye, according to a definite law, is enlarged less than the brain, but especially (for about $^3/_4$ of the calculated surface dimension) that in Pithecanthropus the latter has double the volume of the brain of an Anthropoid Ape of the same bulk. Accordingly this difference does not entail another brain development.

There is on the other hand a great difference — and a difference of great importance — between the profile of the endocranial cast and that of the Neandertal Man of La Chapelle-aux-Saints. Though in consequence of its comparatively small height this seems more simean than that of Homo sapiens, also this human brain profile is seen to rise considerably above that of Pithecanthropus and Hylobates. From the front to the occiput, but the greatest difference, both in Homo neandertalensis and in Homo sapiens, is in the parietal region, near which apex the upper end of the sulcus centralis is situated. Hence the human brain may be called **macrotactile** in the highest degree, as occupying the highest stage of the development on the tactilomotive basis of the central gyri of the cerebral hemispheres characteristic of the Primates. Undoubtedly because the hand, the unequalled tactile organ and the most efficacious instrument of the entire animal world, reaches its highest perfection in Man.

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[1]) The transversal plane described in These Proceedings Vol. XXIII, (1921), p. 1272.

The varying ratio of the size of the brain and the eyes may also account for the displacement of the bregma-point in a frontal direction, from Hylobates to Pithecanthropus, and from the latter to Homo neandertalensis, the lambda-point continuing to occupy about the same place in the side view outlines. The increasing size of the parietal bone appearing from this and its encroaching on the frontal bone, is apparently only a consequence of the orbital impression in the frontal lobe of the brain which becomes relatively smaller, and extends therefore less far backwards.

In Pithecanthropus the bregma-point lies about 12 mm. before the sulcus centralis of the brain, but still considerably behind the sulcus praecentralis superior. Just as in the Hylobatidae also the upper part of the coronal suture lies *behind* the sulcus praecentralis superior, in contrast with the large Anthropoid Apes and with Man, in which latter the crown suture is situated on an average from 2 to 3 cm. *before* the sulcus praecentralis.

Hence the coronal suture may be said to rise steeper and steeper with increasing size of the body; in the Hylobatidae and Pithecanthropus this is, however, attended with only slight displacement of the suture with respect to the gyrus centralis anterior, whereas this displacement is very considerable in the large Anthropoids and in Man.

That at least the displacement of the sulcus centralis (which is certainly a cytotectonic and physiological boundary line at the cerebral surface of the Primates) may be accounted for by the ratio of the size of the eye and the brain varying with the size of the body, is proved by the fact that the ratio of the parts of the surface of the brain lying before and behind this sulcus remains the same in

Pithecanthropus erectus and Hylobates agilis, viz. 1:1.73, measured to morphologically the same transversal plane.

In the frontal region of the cerebral hemispheres the gyri can be very clearly distinguished at the endocranial cast, as was already said above. They are slightly less simple on the left side than on the right, where they have also been preserved over a somewhat larger extent. For an immediate orientation the central and precentral sulci and the Sylvian fissure can at once be easily recognized.

Most conspicuous, to the front, is, on the right side, the **sulcus frontalis inferior,** as clear and unmistakable as in any human hemisphere, but in the simplest form, which it presents shortly before birth.

It has the shape of a ↄ lying almost on its side, the lower half of which encompasses a strong front branch of the Sylvian fissure. On comparison of this endocranial cast with endocranial casts of Chimpanzee, Gorilla, and Gibbons the validity of the interpretation of Chudzinsky, Eberstaller, Hervé, and Waldeyer appears with the greatest clearness, according to which the sulcus frontalis inferior of Man is homologous with the sulcus fronto-orbitalis of the Apes and the microcephalics, and then also the sulcus principalis or rectus of the Apes homologous with the sulcus frontalis medius plus fronto-marginalis (Wernicke) of Man. In consequence of the greater increase of the brain volume compared with the eye, the sulcus fronto-orbitalis is seen to shift from the orbital to the lateral surface of the hemisphere from Hylobates to Chimpanzee (and Gorilla) for a great part, and to Pithecanthropus entirely. The cerebral convolution lying under and behind the sulcus fronto-orbitalis of the Apes is, therefore, the gyrus frontalis inferior. The conclusive establishment of these homologies is certainly the most important fact taught us by the unequalled endocranial cast of the Trinil calvarium. We meet, therefore, already with perfectly human forms in the frontal cerebral gyri of Pithecanthropus, and these forms are fundamentally the same as those possessed by the Hylobatidae, which we may admit to resemble the general ancestor of the Simiidae or Manlike Apes, notwithstanding their specialized features. The fundamental plan of the human brain thus evidently dates from the primitive simian one.

The two knees or genua of the sulcus centralis, so characteristic of Manlike Apes and constantly occurring, the upper or cruro-brachial and the lower or brachio-facial genu, physiological cortex boundaries according to Sherrington, are also met with in Pithecanthropus, and here the great length of the motor cortex region for the leg, at the expense of that for the arm, suggest a human gait and posture of body. For the rest the brain of Pithecantropus is not distinguished qualitatively, only quantitatively, from that of the Anthropoid Apes.

The **double brain quantity** (for equal bulk), is the most important characteristic that distinguishes Pithecantropus from the Anthropoid Apes, though in this respect it is still far inferior to Man, who (calculated for equal body weight) possesses four times the brain quantity of the latter. The laws of the relation between the weight of brain and body of related animal genera also teach that a higher organisation is obtained by a doubling of the whole or a very large part — the half — of the brain quantity (calculated for equal body-weights), evidently by cell-division, and resulting in a twofold, fourfold, or a sesquialteral, threefold, sixfold increase.

The same doubling of the whole brain quantity as from the simian level to that of Pithecanthropus and from Pithecanthropus to Man is found in the American ape genera Callithrix to Saimiri and Saimiri to Cebus, the generalized Ungulate Tragulus to the specialized modern types of Ruminantia, Mus to Lepus, Putorius to Mustela, Sorex to Talpa. In the same quadruple relation as Man and the Manlike Apes are Cebus (also Ateles) and Callithrix, Tupaia and Centetes. The Simiidae have once and a half times the brain quantity of the Cynopithecidae. Sciurus has the triple brain quantity of Mus, and the Megachiroptera the triple of the Microchiroptera, the same as Pithecanthropus in relation to the Cynopithecidae. Elephas has the sixfold quantity (always calculated for the same body-weight) of Procavia and Moeritherium. In the same relation is Man to the Cynopithecidae.

It seems to me that it is evident, at least, from all this that Man and Pithecanthropus, both descend from a common primitive Simian ancestor. From this among the living species, the Hylobatidae, though greatly differentiated by their long arms and sabre-shaped canines, depart least, several fossil Simiidae still less.

Also through his **mandible and teeth** Pithecanthropus deviated less from this common stock type than the three living Gigantanthropoidea and the Hylobatidae.

Besides the calvarium and the left femur, three different teeth were dug up at Trinil, and nearly a year before the discovery there of the first fossil remain of Pithecanthropus (the hindmost right upper molaris), a **mandibular fragment,** a small piece on the right of the symphysis, was found in the same Kendeng-layers, but at 40 km. distance on the E.S.E. of Trinil, namely at Kedung

Brubus, among other fossil remains of the Kendeng-fauna. Its specific gravity is the same as that of the teeth and the other remains of Pithecanthropus. A brief description of the mandibular fragment appeared at Batavia in 1891, in the "Verslag van het Mijnwezen", and I mentioned it, in a few words, in the "Natuurkundig Tijdschrift voor Nederlandsch-Indië" of the same year. I then considered it a remain of a not exactly determinable human species, "of another and probably lower type" than those existing and the extinct European diluvial species. This particularly on the ground of a peculiarity in connection with the place of attachment of the digastric muscle.

The mandibular fragment is a scalene-triangular piece of the corpus mandibulae, with as basis 36 mm. (measured rectilinearly) of the lower border, immediately on the right of the symphysis. The apex is formed by the root of the anterior praemolar tooth, which root has been preserved for the greater part. It is there 30 mm. high. There further is preserved the back half of the flat alveolus of the caninus with its root point and part of the front plane of the alveolus of the posterior praemolar tooth, under which is situated the front edge of the foramen mentale, 12 mm. above the sharp lower border. In its full thickness the corpus mandibulae has only remained preserved at the septum of the alveoli of the caninus and the anterior praemolar tooth.

I now ascribe also this mandibular fragment to Pithecanthropus erectus, because what the teeth teach us is quite corroborated by the morphological characters of this small, but all the same very significant piece of the mandible. The three teeth of Trinil are the upper left second and right third molar tooth, and the lower left anterior premolar tooth. The root of the lower anterior premolar tooth, which has remained preserved in the mandible fragment, closely resembles the root of that tooth of Trinil, and the alveolus still present in front of it, with the root-point of the canine tooth, betrays comparatively small canini, a caninus of the maxilla in keeping with this short premolar crown of the mandible. In view of this it is highly probable that the mandible fragment of Kedung Brubus derives from (another individual of) Pithecanthropus erectus, an at any rate rare species of the Kendeng-fauna.

The jaw and the teeth appear to have been almost perfectly human in their front part, particularly in the shape of the symphysis and the lower anterior premolar tooth and the canini, all of which differ so characteristically from those of the Manlike Apes. Crown and root of the premolar tooth are mesio-distally narrow, and the root, which has a double formation, is only divided near the point. Also the alveolus of the lower caninus is flattened in the said direction.

Very remarkable is at the mandible fragment the extensive, broad, and long area of attachment of the digastric muscle, which makes the under part of the bone angular over a great length (as far as past m^2). This attachment of the digastric muscle which, as in the gibbons, extends far backward, is incompatible with a function of the tongue as an organ of speech. The muscle may, indeed, have been particularly powerful (much stronger in proportion to the size of the body than in the gibbons), because it had to bear a comparatively greater weight, on account of the erect attitude of Pithecanthropus.

Very different from that of the Manlike Apes, the symphysis must have been quite human. In the large Anthropoid Apes a plane applied at right angles to the alveolar line, between the caninus and the anterior premolar, always cuts the corpus mandibulae at a considerable distance before the back edge of the symphysis; in the small gibbon species it just strikes the back edge. In the mandibles of Mauer, Spy, and La Naulette, as well as in modern Man, on the contrary, this plane remains about 1 cm. behind the symphysis. The plane in question generally passes right through the septum of the alveoli of the teeth mentioned, and also in our fossil mandibular fragment this plane remained far (7 mm. according to estimation) behind the symphysis.

The root of the **anterior premolar** tooth in this fossil is broad, bucco-lingually 8.4 mm., and flattened in mesio-distal direction to 5.0 mm. Hence it has about the same dimensions as the root of the homonymous tooth of Trinil : 8.1 and 4.2 mm. Like this, it is composed of two root elements placed almost transversally, to be recognized by the two canals. The two teeth were evidently of the same type.

This tooth of Trinil was very human, as also appears from the diameters of the crown: 8.2 mm. bucco-lingual, 7.0 mm. mesio-distal, breadth index 117. It is undoubtedly an *anterior* premolar tooth, for the buccal cusp is much larger than the lingual one, hardly deserving the name of cusp, which feature makes the masticatory surface oblique, the buccal surface is strongly bent inward, bulging outward and much higher than the lingual surface; besides the middle crista is of a type only met with at anterior praemolar teeth of Anthropoids. In front and at the back the crown presents a facet of contact, with the canine tooth and the posterior premolar tooth of the lower jaw, and at the upper edge of the buccal surface a facet of wear with the caninus of the upper jaw. The flat root of this tooth consists of a front-outer and a back-inner element. At the back-side they are separated only at the point of the former (the other point is broken off), over a length of 3.6 mm., but at the front side the disjunction of the root extends, as a deep fissure, 8 mm. more towards the crown. The total length of the front-outer root-element is 20.1 mm. The two roots of the homonymous tooth of the Anthropoid Apes, which are entirely or for the greater part separated and placed almost perfectly behind each other, are found here, as it were, from front backwards pressed almost crosswise and grown together.

The two **upper molar teeth** of Trinil, both with a triple disposition of the buccal roots, which have, however, been fused, evidently derive from one and the same individual. The rugosity of the crown of m^3 is much greater than in the Orang-utan, though in the shape of its crown this tooth presents a striking resemblance with some orang-utan teeth which I collected in caves in Central Sumatra. Besides a shape of crown as of m^2 is frequently met with in the Orang-utan. The slight development in both crowns of the hindmost buccal cusp, and the strong divergence of the roots may be in connection with the trigonocephalism of the cranium, which promoted the growth of the teeth in transversal direction above that in sagittal direction. The brachycephalism of the Orang-utan and the trigonocephalism of this individual Pithecanthropus have here the same result. For the rest, the circumstance that the three teeth were found at Trinil in exactly the same plane of the andesite tufa, with and comparatively near other remains of Pithecanthropus, renders the probability that they originate from one individual almost a certainty.

The crowns of the molar teeth have these measures, (mm.):

	bucco-lingual	mesio-distal	breadth index
m^2	13.8	12.0	115.0
m^3	15.3	11.3	135.4

As to size, m^3 is greatly exceeded by some orang-utan teeth from caves of Central Sumatra. Two of these attain bucco-lingually 18.7 and 20.2 mm. and mesio-distally 14.0 and 15.5 mm.; an m^3 thus reaches 19.0 and 17.7 mm.

The length of the two roots of the two molar teeth is as follows: the buccal of m^2 13 mm. and of m^3 14 mm., the lingual of m^2 12 mm. and of m^3 12.5 mm. They diverge greatly (the lingual deviates most from the vertical); between the outside of the points the distance is 16.5 mm. in m^2 and 18.0 mm. in m^3. The lingual roots are simple, the buccal ones on the other hand are composed, as stated above, in both teeth, of *three* fused elements: two outer (buccal) root elements, and one at the inside of the foremost of the latter. In Man as well as in the Anthropoid Apes there are found as a rule one lingual and two buccal roots. The fusion of the two buccal roots points again to the mesio-distal shortening of the dentition, the incipient splitting up of the mesio-buccal root to the broadening in lingual-buccal direction. These are certainly individual characters, which establish that the two molar teeth belong individually together. The peculiar individual growth of the two teeth is, as was already said above, evidently in connection with the trigonocephalism, which caused the cranium to widen towards the bottom in its front half. The direction of the roots backwards may, possibly, have the same meaning, for the dental root grows towards the point, and there was more room towards the back.

The approach of the mandible and the teeth, as also of the femur, to the human type, and the large cranial capacity, added to considerations on the brain-quantities in nearly allied mammalian genera, all this leads me to the conclusion that Pithecanthropus should be considered as a member, but a distinct genus, of the family of the Hominidae.

Chemistry. — *"In-, mono- and plurivariant equilibria"*. XXVI. By Prof. F. A. H. SCHREINEMAKERS.

(Communicated at the meeting of March 29, 1924).

Equilibria of n components in r phases.

We take into consideration an equilibrium E and other equilibria, the phases of which differ unfinitely little from those of E. When we express the compositions of the phases in composants, then we may choose them arbitrarily, so that we may take for this also one or more of the phases of the equilibrium E itself. As, therefore, those composants are really also phases of the equilibrium E, we shall call them "real" composants; the other composants, which consequently still rest arbitrary, we call the "free" ones.

We now assume that at a temperature T_0 and under a pressure P_0 an equilibrium

$$E_0\,(n\,.\,r) = L + M + N \ldots + R \quad . \quad . \quad . \quad . \quad . \quad (1)$$

exists of n components in r phases. In order to represent the phases of an equilibrium:

$$E\,(n\,.\,r) = F_1 + F_2 \ldots + F_r \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which differ from E_0 $(n\,.\,r)$, we must choose n composants. For this we take the r phases $L\,M \ldots$ of the equilibrium E_0 $(n\,.\,r)$, consequently real composants; further we take the $n-r$ free composants $X\,Y \ldots$; of course one or more of those may also be components. When we represent an arbitrary phase F of the equilibrium $E\,(n\,.\,r)$ by:

$$F = xX + yY \ldots + lL + mM \ldots + qQ + \varrho R \ldots \quad , \quad (3)$$

wherein $\varrho = 1 - x - y \ldots - l \ldots - q$, then we may represent F_1 by giving to all variables in (3) the index 1; F_2 by giving them the index 2, etc.

We now assume that the phases of $E\,(n\,.\,r)$ differ infinitely little from those of $E_0\,(n\,.\,r)$, viz. F_1 from L, F_2 from $M \ldots$ and F_r from R. Then it appears from (1) and (3) that all variables in (3) should become infinitely small, excepted $l_1, m_2, n_3 \dots q_{r-1}$ which approach to zero.

When we call the thermodynamical potentials of $F_1 F_2$ etc. $\zeta_1 \zeta_2$ etc. and when we represent by A the form:

$$\zeta - x\frac{\partial \zeta}{\partial x} - y\frac{\partial \zeta}{\partial y} \dots - l\frac{\partial \zeta}{\partial l} - m\frac{\partial \zeta}{\partial m} \dots - q\frac{\partial \zeta}{\partial q} \quad . \quad . \quad . \quad (4)$$

then we find for an arbitrary temperature T and pressure P the $r-1$ equations of equilibrium:

$$A_1 = A_2 = \dots = A_r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

We obtain $A_1 A_2$ etc. from A by giving in (4) the indices 1, 2 etc. to all variables (ζ also). Of course those equations are valid as well for finite as for unfinitely small values of the variables. Further we still get the $(r-1)(n-1)$ conditions for equilibrium:

$$\left.\begin{aligned} \frac{\partial \zeta_1}{\partial x_1} = \frac{\partial \zeta_2}{\partial x_2} = \dots = \frac{\partial \zeta_r}{\partial x_r} \\ \frac{\partial \zeta_1}{\partial y_1} = \frac{\partial \zeta_2}{\partial y_2} = \dots = \frac{\partial \zeta_r}{\partial y_r} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and the other ones, which follow herefrom, by substituting x or y by $z\, u \dots l\, m \dots$ and q.

When we include in the calculation T and P, then consequently we have $(n-1)\,r+2$ variables and $n(r-1)$ equations; therefore the equilibrium $E(n\,.\,r)$ has $n-r+2$ freedoms. When we put in (5) and (6) $l_1 = 1$, $m_2 = 1$, $n_3 = 1$ and $q_{r-1} = 1$ and further all other variables equal to zero, then we find the conditions for equilibrium for the equilibrium $E_0(n\,.\,r)$; when we put $l_1 = 1 + \Delta l_1$, $m_2 = 1 + \Delta m_2 \dots q_{r-1} = 1 + \Delta\, q_{r-1}$ and when we take for the other variables infinitely small values, then we find the equations for equilibrium for the equilibrium $E(n.r)$ at $T_0 + \Delta T$ and $P_0 + \Delta P$. From (5) follow the $r-1$ equations:

$$\Delta\, A_1 = \Delta\, A_2 = \dots = \Delta\, A_r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Further follow from (6) the $(r-1)(n-1)$ equations:

$$\Delta\frac{\partial \zeta_1}{\partial x_1} = \Delta\frac{\partial \zeta_2}{\partial x_2} = \dots = \Delta\frac{\partial \zeta_r}{\partial x_r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

etc. Herein the sign Δ means that we must take all increases, if necessary those of higher order also. It now follows from (4) that (7) may be satisfied by:

$$\left.\begin{aligned}
\Delta A_1 &= -\eta_1\,\Delta T + v_1\,\Delta P - \Delta\frac{\partial\zeta}{\partial l} - \frac{1}{2}K_1 \\
\Delta A_2 &= -\eta_2\,\Delta T + v_2\Delta P - \Delta\frac{\partial\zeta}{\partial m} - \frac{1}{2}K_2 \\
&\cdots\cdots\cdots\cdots\cdots \\
\Delta A_{r-1} &= -\eta_{r-1}\,\Delta T + v_{r-1}\,\Delta P - \Delta\frac{\partial\zeta}{\partial q} - \frac{1}{2}K_{r-1} \\
\Delta A_r &= -\eta_r\;\;\Delta T + v_r\;\;\Delta P - \frac{1}{2}K_r
\end{aligned}\right\} \quad . \quad (9)$$

wherein we may give to $\Delta\frac{\partial\zeta}{\partial l}$, $\Delta\frac{\partial\zeta}{\partial m}$ etc. in accordance with (8) each of the indices from 1 to r. Further is

$$K = \left(x\frac{\partial\zeta}{\partial x} + y\frac{\partial\zeta}{\partial y} + \ldots\right)^{(2)}.$$

We now shall mean by the region of an equilibrium: the collection of all complexes, which arise when we add to one another the phases of that equilibrium in all possible ratio's. In a graphical representation we may imagine such a region to be represented by a collection of points. The region of an equilibrium $E(n\,.\,2)$ is, therefore, a straight line, which unites the two phases, namely the part, which is situated between both phases. The region of the equilibrium $E(n\,.\,3)$ is the triangle, which has as angle-points those three phases, etc.

We now may put the question: the regions of two equilibria $E_0(n\,.\,r)$ and $E(n\,.\,r)$ may they have common points at the same temperature and pressure?

In order to represent the region of the equilibrium:

$$E(n\,.\,r) = F_1 + F_2 + \ldots + F_r\,,$$

we must take in the complex

$$a_1\,F_1 + a_2\,F_2 + \ldots + a_r\,F_r \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$a_1\,a_2$ etc. positive and we must give to their ratio's all possible values. In the complex (10) none of the free composants $X\,Y \ldots W$ may occur, in order that this region has one or more points in common with the region of $E_0(n\,.\,r)$. It now follows from (10) that it must be possible that is satisfied:

$$\left.\begin{aligned}
\Sigma\,(ax) &= a_1\,x_1 + a_2\,x_2 + \ldots + a_r\,x_r = 0 \\
\Sigma\,(ay) &= a_1\,y_1 + a_2\,y_2 + \ldots + a_r\,y_r = 0 \\
&\cdots\cdots\cdots\cdots\cdots \\
\Sigma\,(aw) &= a_1 w_1 + a_2\,w_2 \;\ldots\; + a_r\,w_r = 0
\end{aligned}\right\} \quad . \quad . \quad . \quad (11)$$

As we take under consideration the equilibria $E_0(n.r)$ and $E(n.r)$ at the same P and T, we must put in (9) $\Delta T = 0$ and $\Delta P = 0$. Hence it follows:

$$\Delta\frac{\partial\zeta}{\partial l}=0 \qquad \Delta\frac{\partial\zeta}{\partial m}=0\ldots\Delta\frac{\partial\zeta}{\partial q}=0 \quad . \; . \; . \; . \quad (12)$$

The stability requires that the magnitudes K_1 K_2 etc. from (9) are positive. As the terms, wich are equal to zero, in accordance with (12), disappear herefrom, we may write:

$$K = x\,\Delta\frac{\partial\zeta}{\partial x} + y\,\Delta\frac{\partial\zeta}{\partial y} + \ldots w\,\Delta\frac{\partial\zeta}{\partial w} . \quad . \; . \; . \; . \quad (13)$$

When we add K_1 K_2 etc., after having multiplied them successively with a_1 a_2 etc., then follows:

$$\Sigma\,(a\,x)\,.\,\Delta\frac{\partial\zeta}{\partial x} + \Sigma\,(a\,y)\,.\,\Delta\frac{\partial\zeta}{\partial y}\ldots + \Sigma\,(a\,w)\,.\,\Delta\frac{\partial\zeta}{\partial w} \quad . \; . \quad (14)$$

which form must be positive. Consequently it is impossible to satisfy the equations (11), as (14) should then be equal to zero. Consequently we find:

the regions of the stable equilibria $E_0(n.r)$ and $E(n.r)$ may never have points in common at the same temperature and under the same pressure.

The conjugation-line of the two phases of an equilibrium $E_0(n.2)$, therefore, never intersects the conjugation-line of the equilibrium $E(n.2)$; the three-phases-triangle of an equilibrium $E_0(n.3)$, therefore, never intersects the three-phases-triangle of the equilibrium $E(n.3)$, etc. In the previous communication we have deduced this property for a special case, viz. for an equilibrium $E(3.2)$ in quite another way. We also easily find this property in the following way. Suppose the regions of the equilibria $E_0(n.r)$ and $E(n.r)$ have a point in common; then this point may represent either a complex of the phases of the equilibrium E_0 or one of the phases of the equilibrium E. As T and P are, however, the same for both complexes, the complex with the largest thermodynamical potential shall be converted into that with the smallest one. Consequently both complexes cannot exist at the same time or in other words, the two regions cannot have a point in common.

The deductions above are no more valid when we keep no more constant either the temperature or the pressure, or both. Then we find:

the regions of the equilibrium $E_0(n.r)$ at T_0 and of the equilibrium

$E(n.r)$ at $T_0 + \Delta T$ have points in common, also when the pressure for both is the same.

We shall represent an equilibrium of n components in r phases, when a phases-reaction is possible between those r phases, by $E(n.r)(r)$. We now assume that at T_0 and P_0 an equilibrium

$$E_0(nr)(r) = L + M + N \ldots + Q + R \quad . \quad . \quad . \quad . \quad (15)$$

exists; we represent the phases-reaction by

$$\lambda_1 L + \lambda_2 M \ldots + \lambda_{r-1} Q + \lambda_r R = 0 \quad . \quad . \quad . \quad . \quad (16)$$

As, in accordance with (16) those phases are dependent on one another, we may take only $r-1$ of them as composants; for this we take $MN \ldots R$. We now represent an arbitrary phase F in stead of by (3) by

$$F = xX + yY \ldots + mM + nN \ldots qQ + \varrho R \quad . \quad . \quad (17)$$

viz. by $r-1$ real and $n-r+1$ free composants. The phases of an arbitrary equilibrium:

$$E(n.r) = F_1 + F_2 \ldots + F_r \quad . \quad . \quad . \quad . \quad . \quad (18)$$

we represent again by giving in (17) the indices $1, 2, \ldots r$ to the variables. When we represent again by A

$$\zeta - x\frac{\partial \zeta}{\partial x} - y\frac{\partial \zeta}{\partial y} \ldots - m\frac{\partial \zeta}{\partial m} \ldots - q\frac{\partial \zeta}{\partial q} \quad . \quad . \quad . \quad (19)$$

then for the equilibrium $E(n.r)$ at arbitrary P and T the $r-1$ equations

$$A_1 = A_2 = \ldots = A_r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

are following again and $(r-1)(n-1)$ equations like in (6).

We now assume again that the phases of the equilibrium $E(n.r)$ differ infinitely little from those of $E_0(n.r)(r)$. When the composition of F_1 must approach to L, then it appears from (17) that we must give to the quantities $x_1\,y_1$ etc. of the free composants the infinitely small values $\Delta x_1\; \Delta y_1$ etc. In connection with the composition of L, which is following from (16) it is apparent that we must put:

$$m_1 = -\frac{\lambda_2}{\lambda_1} + \Delta m_1 \qquad n_1 = -\frac{\lambda_3}{\lambda_1} + \Delta n_1 \ldots q_1 = -\frac{\lambda_{r-1}}{\lambda_1} + \Delta q_1$$

wherein Δm_1 etc. are infinitely small.

When the composition of F_2 approaches to M, then we must put $m_2 = 1 + \Delta m_2$ and further we must replace all other variables $x_2\,y_2$ etc. by the infinitely small values $\Delta x_2\,\Delta y_2$ etc. For the other phases $F_2 \ldots F_r$ we find that all variables must be infinitely small, excepted

$$n_2 = 1 + \Delta n_2 \ldots q_{r-1} = 1 + \Delta q_{r-1}.$$

In order to satisfy (20) we now have, as follows from (19):

$$\left.\begin{aligned}
\lambda_1 \Delta A_1 &= - \lambda \ \eta_1 \Delta T + \lambda_1 v_1 \Delta P + \lambda_2 \Delta \frac{\partial \zeta}{\partial m} + \lambda_3 \Delta \frac{\partial \zeta}{\partial n} \ldots \\
&\qquad \ldots + \lambda_{r-1} \Delta \frac{\partial \zeta}{\partial q} - \frac{1}{2} \lambda_1 K_1 \\
\Delta A_2 &= - \eta_2 \Delta T + v_2 \Delta P - \Delta \frac{\partial \zeta}{\partial m} - \frac{1}{2} K_2 \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
\Delta A_{r-1} &= - \eta_{r-1} \Delta T + v_{r-1} \Delta P - \Delta \frac{\partial \zeta}{\partial q} - \frac{1}{2} K_{r-1} \\
\Delta A_r &= - \eta_r \Delta T + v_r \Delta P - \frac{1}{2} K_r
\end{aligned}\right\} \quad . \ . \quad (21)$$

wherein again

$$K = \left(\Delta x . \frac{\partial \zeta}{\partial x} + \Delta y . \frac{\partial \zeta}{\partial y} + \ldots \right)^{(2)}.$$

Further we may give, as it shall appear at once, to $\Delta \dfrac{\partial \zeta}{\partial m}$ etc. all indices from 1 to r. We yet have viz. the $(r-1)(n-1)$ equations (6) from which follows:

$$\Delta \frac{\partial \zeta_1}{\partial x_1} = \Delta \frac{\partial \zeta_2}{\partial x_2} = \ldots = \Delta \frac{\partial \zeta_r}{\partial x_r} \quad . \ . \ . \ . \quad (22)$$

and the other ones, which we get herefrom by substituting x by $y \, z \ldots m \ldots q$.

When we add the equations (21) after having multiplied ΔA_2 with λ_2, ΔA_3 with λ_3 etc., then we get, as $\Delta A_1 = \Delta A_2 = \ldots = \Delta A_r$, and as $\Sigma(\lambda)$ must be zero in accordance with (16):

$$- \Sigma (\lambda \eta) . \Delta T + \Sigma (\lambda v) . \Delta P = \frac{1}{2} \Sigma (\lambda K) \quad . \ . \ . \quad (23)$$

Further from (21) still the $r-2$ equations follow:

$$\left.\begin{aligned}
(\eta_r - \eta_2) \Delta T + (v_2 - v_r) \Delta P &= \Delta \frac{\partial \zeta}{\partial m} \\
\ldots\ldots\ldots\ldots\ldots\ldots & \\
(\eta_r - \eta_{r-1}) \Delta T + (v_{r-1} - v_r) \Delta P &= \Delta \frac{\partial \zeta}{\partial q}
\end{aligned}\right\} \quad . \ . \quad (24)$$

wherein K_2 etc. have been omitted, as they are infinitely small with respect to the terms, which are already written down.

Consequently we have in (22), (23) and (24) $rn - n$ equations between the $rn - r + 2$ variables for definition of the equilibrium $E(n.r)$ at $T_0 + \Delta T$ and under $P_0 + \Delta P$.

Firstly we now put the question: when an equilibrium $E_0(nr)(r)$ exists at T_0 and under P_0, still other equilibria $E(n.r)$ may they exist at the same temperature and under the same pressure?

As the equilibrium now has still $n - r$ freedoms only, and as r must be 2 at least, this case may, therefore, occur only with systems with 3 or more components. As further we must put $\Delta T = 0$ and $\Delta P = 0$, it follows from (23):

$$\lambda_1 \left(\Delta x_1 \frac{\partial \zeta_1}{\partial x_1} + \Delta y_1 . \frac{\partial \zeta_1}{\partial y_1} \dots \right)^{(2)} + \lambda_2 \left(\Delta x_2 . \frac{\partial \zeta_2}{\partial x_2} + \Delta y_2 \frac{\partial \zeta_2}{\partial y_2} \dots \right)^{(2)} + \dots = 0 . \quad (25)$$

The stability requires that each of the forms in parenthesis is positive. Of course the increases in (25) are not independent on one another, but they are united by the equations (22) and (24), wherein $\Delta T = 0$ and $\Delta P = 0$. Hence it appears, therefore, that it shall depend on many conditions whether (25) may be satisfied or not. When this is not the case, then at T_0 and under P_0 only the equilibrium $E_0(nr)(r)$ exists; when it is really the case then still also other equilibria $E(n.r)$ exist, which then have $n - r$ freedoms. Later we shall illustrate this by treating an example.

Let us assume that (22), (24) and (25) may be satisfied by definite values of the increments Δx_1 Δy_1 etc. Then a definite equilibrium $E(n.r)$ exists. However, the equations may be satisfied also by giving the opposite sign to all increments; consequently also an other equilibrium $E^1(n.r)$ exists. We shall call two equilibria, which satisfy this condition the reflected images of one another, because, in a graphical representation, they are looking somewhat alike. Consequently we may say, also in connection with the previous:

when at T_0 and under P_0 besides the equilibrium $E_0(nr)(r)$ still other equilibria $E(n.r)$ exist, then they are two by two the reflected image of one another; the regions of those equilibria have no points in common.

We now may also answer the question: when an equilibrium $E_0(nr)(r)$ exists at T_0 and under P_0, which other equilibria $E(n.r)$ are existing then under the same pressure but at the temperature $T_0 + \Delta T$?

In order to answer this question we put $\Delta P = 0$; (23) becomes then:

$$\Sigma(\lambda\eta).\Delta T = -\frac{1}{2}\Sigma\lambda\left(\Delta x.\frac{\partial\zeta}{\partial x}+\Delta y.\frac{\partial\zeta}{\partial y}\dots\right)^{(2)} \quad . \quad . \quad (26)$$

The equations (22) and (24) are homogeneous and of the first degree with respect to ΔT and the other increments. We may satisfy those equations by taking all increments of the same order, but ΔT of the order $(\Delta x)^2$. In the equations (22) and (24) at first approximation all terms with ΔT may disappear. Then it follows from (24):

$$\Delta\frac{\partial\zeta}{\partial m}=0 \qquad \Delta\frac{\partial\zeta}{\partial n}=0\dots\Delta\frac{\partial\zeta}{\partial q}=0 \quad . \quad . \quad . \quad . \quad (27)$$

with the indices 1 to r. We may use those equations in order to simplify (26).

$\Sigma(\lambda\eta)$ in (26) is the increase of entropy when in the equilibrium $E_0(n.r)(r)$ the phases-reaction (16) occurs. We imagine that the reaction proceeds in that direction, at which the entropy increases; then $\Sigma(\lambda\eta)$ is positive and the signs of λ_1 λ_2 etc. are defined.

We now distinguish three cases.

1. the second part of (26) is always positive.

Then (26) can only be satisfied by giving a positive value to ΔT. Hence follows:

equilibria $E(n.r)$ exist only at temperatures higher than T_0; or also: the temperature T_0 of the equilibrium $E_0(nr)(r)$ is a minimum-temperature for the equilibria $E(n.r)$.

2. The second part of (26) is always negative.

We find: equilibria $E(n.r)$ exist only at temperatures lower than T_0; or also: the temperature T_0 of $E_0(nr)(r)$ is a maximumtemperature for the equilibria $E(n.r)$.

3. The second part of (26) may be positive, negative and zero.

We find: equilibria $E(n.r)$ exist as well at T_0 as at higher and lower temperatures; the temperature T_0 of the equilibrium $E_0(n.r)(r)$ is nor maximum- nor minimum-temperature for the equilibria $E(n.r)$.

Just as above we find that the equilibria existing at $T_0+\Delta T$ (ΔT positive, negative or zero) are two by two the reflected image of one another and that their regions have no points in common.

As the question: when at T_0 and under P_0 an equilibrium $E_0(nr)(r)$ exists, which equilibria $E(n.r)$ exist then at the same temperature but under a pressure $P_0 + \Delta P$, leads to the same results, we shall not discuss this question.

We now shall put the question: when at T_0 and under P_0 exists an equilibrium $E_0 (n\,r)(r)$, another equilibrium $E(n\,r)(r)$ can it exist at $T_0 + \Delta T$ and under $P_0 + \Delta P$ and under which conditions?

As in the equilibrium $E_0 (n\,r)(r)$ the phases-reaction (16) occurs, must be:

$$\Sigma(\lambda) = \lambda_1 + \lambda_2 + \ldots + \lambda_r = 0 \quad . \quad . \quad . \quad . \quad . \quad (27)$$

In accordance with our assumption, also a phases-reaction may occur in the equilibrium $E(n\,r)(r)$; we represent this by:

$$\lambda'_1 F_1 + \lambda'_2 F_2 \ldots + \lambda'_r F_r = 0 \quad . \quad . \quad . \quad . \quad . \quad (28)$$

In accordance with (17) and (18) it must be possible to satisfy:

$$\Sigma(\lambda' x) = 0 \qquad \Sigma(\lambda' y) = 0 \ldots \Sigma(\lambda' m) = 0 \ldots \Sigma(\lambda' q) = 0 \quad (29)$$

As the phases of $E(n\,r)(r)$ and $E_0 (n\,r)(r)$ differ infinitely little in composition, they pass into:

$$\left.\begin{array}{l} \Sigma(\lambda' . \Delta x) = 0 \qquad \Sigma(\lambda' . \Delta y) = 0 \ldots \\ \Sigma(\lambda' m) = -\lambda'_1 \dfrac{\lambda_2}{\lambda_1} + \lambda'_2 + \Sigma(\lambda' \Delta m) = 0 \\ . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ \Sigma(\lambda' q) = -\lambda'_1 \dfrac{\lambda_{r-1}}{\lambda_1} + \lambda'_{r-1} + \Sigma(\lambda' \Delta q) = 0 \end{array}\right\} \quad . \quad (30)$$

From this we find that at first approximation the $n-r+1$ equations must be satisfied:

$$\left.\begin{array}{l} \Sigma(\lambda . \Delta x) = \lambda_1 \Delta x_1 + \lambda_2 \Delta x_2 \ldots + \lambda_r \Delta x_r = 0 \\ . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ \Sigma(\lambda . \Delta w) = \lambda_1 \Delta w_1 + \lambda_2 \Delta w_2 \ldots + \lambda_r \Delta w_r = 0 \end{array}\right\} \quad . \quad . \quad (31)$$

wherein $\lambda_1\, \lambda_2$ etc. have the values from (16). Those $n-r+1$ conditions contain only increments, which refer to the quantities of the free composants. We now have for the definition of the equilibrium $E(n\,r)(r)$ at $T_0 + \Delta T$ and under $P_0 + \Delta P$ the $r\,n-n$ equations (22), (23) and (24) and further the $n-r+1$ equations (31), consequently $r\,n-r+1$ equations between the $n\,r-r+2$ variables. The equilibrium $E(n\,r)(r)$ has, therefore, one freedom.

We cannot satisfy those equations by taking ΔT or ΔP or both of higher order than the other increases, but we may satisfy them by taking all increments of the same order.

Then it follows from (23):

$$\frac{\Delta P}{\Delta T}=\frac{\Sigma(\lambda\eta)}{\Sigma(\lambda v)} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (32)$$

Consequently to each change of temperature ΔT belongs a change in pressure ΔP, defined by (32). We find, therefore: when an equilibrium $E_0(n\,r)(r)$ exists at T_0 and under P_0, then at $T_0+\Delta T$ also an equilibrium $E(n\,r)(r)$ exists, of which the composition of the phases and the pressure are completely defined (viz. ΔP by 32); also at $T_0-\Delta T$ a similar equilibrium exists; both are the reflected image of one another.

We also may prove that the regions of the equilibria $E_0(n\,r)(r)$ and $E(n\,r)(r)$ have no points in common. When this should be the case, then, as we have seen previously, we must be able to satisfy

$$\Sigma(a\,.\,\Delta x)=0 \qquad \Sigma(a\,.\,\Delta y)=0\;.\,.\,.\;\Sigma(a\,.\,\Delta w)=0\;.\;\;. \quad (33)$$

wherein $a_1\,a_2$ etc. are positive. However, $\Delta x_1\;\Delta x_2$ etc. satisfy the equation (31) in which the coefficients $\lambda_1\lambda_2$ etc. have different signs; consequently they never can satisfy (33).

This is, however, only true for equilibria $E(n\,r)(r)$ in which $r \leq n$. At the deduction is viz. supposed that there is one free composant at least, consequently $n-r+1 \geq 1$ or $r \leq n$. When $r=n+1$ then the two regions have really points in common.

Consequently we may represent in a PT-diagram by a curve an equilibrium $E(n\,r)(r)$. Further we shall divide those equilibria into "transformables" and "intransformables". We shall viz. call the equilibrium transformable when we can proceed a finite part of the PT-curve with a definite complex of the r phases; we call it intransformable when we cannot convert the equilibrium of a temperature T in one of the temperature $T+\Delta T$, unless we add one or more of the components.

It appears from our previous considerations that for $r=n+1$ the equilibria are transformable and for $r<n+1$ intransformable.

A binary equilibrium $E(2\,.\,3)(3)$ f.i. solid + liquid + vapour is, therefore, transformable; we can realise a finite part of its PT-curve with a complex of definite composition.

The binary equilibrium $E(2\,.\,2)(2)$ or in general the equilibrium $E(n\,.\,2)(2)$ of n components f.i. liquid + vapour is intransformable; when we will bring it from a temperature T of its PT-curve towards a temperature $T+\Delta T$, then we must add or remove a little of one or more of the components.

In accordance with our considerations an equilibrium $E(n\,.\,2)\,(2)$ $=$ solid $+$ liquid ought to be intransformable, also when the solid substance has an invariable composition. This is, however, not the case; here the PT-curve is viz. the melting-line of the solid substance and it is clear that we can realise it without it is necessary to change the composition of the complex. Yet this contradiction is only apparent. Solid substance and liquid have viz. the same composition and although they are built up out of n different other substances, yet there is one composant only. Consequently it is not an equilibrium $E(n\,.\,2)\,(2)$ but an equilibrium $E(1\,.\,2)\,(2)$ and it must, therefore, be transformable.

In general an equilibrium $E\,(n\,.\,r)\,(r)$ in which r—1 of the phases have a constant composition and only one of the phases is variable, is an equilibrium $E\,(r—1.\,r)\,(r)$ and consequently transformable.

When an equilibrium $E\,(n\,r)\,(r)$ goes along its PT-curve, then the values of $\Sigma\,(\lambda\eta)$ and $\Sigma(\lambda v)$ change from point to point. When $\Sigma(\lambda\eta)$ becomes zero in a point, then, as is apparent from (32), the tangent in this point is parallel to the T-axis; consequently the pressure in this point is maximum or minimum. In order to examine this further, we must take now into the equations (21) also terms with ΔT^2. Instead of (23) we then find an equation of the form: $\Delta P = \alpha.\, \Delta T^2$. The pressure in this point is, therefore, maximum when $\alpha < 0$ and minimum when $\alpha > 0$.

When in another point $\Sigma\,(\lambda v)$ becomes $=0$, then the tangent in this point is parallel to the P-axis. Instead of (23) we now find: $\Delta T = \beta\,.\,\Delta P^2$. Consequently the temperature is maximum when $\beta < 0$ and minimum when $\beta > 0$.

Therefore, we find: the temperature of an equilibrium $E\,(n\,.\,r)\,(r)$ is maximum or minimum, when the volume does not change at the phases-reaction; the pressure is maximum or minimum, when the entropy does not change, consequently when at the phases-reaction no heat is absorbed or given away.

When we apply this to a binary equilibrium: solid substance $+$ liquid $+$ vapour, consequently to an equilibrium $E(2,\,3)\,(3)$, then follows at once the known property, which defines in this system the position of the point of maximum temperature and pressure.

When an equilibrium $E\,(n.\,r)\,(r)$ goes along its PT-curve, then the ratio's of the reaction-coefficients change from point to point; in definite points then one or more of those coefficients may become zero. When in a point a, consequently at the temp. T_a and under the pressure P_a f.i. λ_1 becomes zero, then reaction (16) passes into:

$$\lambda_2 M + \lambda_3 N \ldots + \lambda_{r-1} Q + \lambda_r R = 0,$$

so that the phase L does not take part into the reaction. The tangent in point a to the PT-curve is defined then by (32) in which, however, the terms $\lambda_1 \eta_1$ and $\lambda_1 v_1$ disappear. We now take away from the equilibrium $E(n.r)(r)$ the phase L, while we let unchanged temperature and pressure; then a new equilibrium $E(n.r-1)(r-1)$ arises. This equilibrium is also represented in a PT-diagram by a curve, which goes of course through the point a. The direction of the PT-curve $E(n.r-1)(r-1)$ in point a is, however, defined also by an equation of the form of (32); herein now also the terms $\lambda_1 \eta_1$ and $\lambda_1 v_1$ are wanted, while the other terms have the same values as for the equilibrium $E(n.r)(r)$. Consequently the two curves touch one another in the point a. As the same discussion is valid also when more coefficients become zero, we get, therefore:

when at T_a and under P_a some phases (f.i. r') of the equilibrium $E(n.r)(r)$ do not participate in the phases-reaction, then the PT-curve of $E(n.r)(r)$ and that of $E(n.r-r')(r-r')$ touch one another in the point a.

In general we may imagine, therefore, $r-1$ equilibria $E(n.r-1)(r-1)$, further $\frac{1}{2}r(r-1)$ equilibria $E(n.r-2)(r-2)$ etc., the PT-curves of which come in contact with that of the equilibrium $E(n.r)(r)$.

When we apply this f.i. to the binary equilibrium $E(2.3)(3) =$ solid substance + liquid + vapour, then herefrom at once the known property follows, that the PT-curve of this equilibrium touches that of the equilibrium $E(2.2)(2) =$ liquid + vapour; that of the equilibrium $E(2.2)(2) =$ solid substance + vapour, consequently the sublimation-line of the solid substance, and that of the equilibrium $E(2.2)(2) =$ solid substance + liquid, consequently the melting-line of the solid substance.

Leiden, Lab. of Inorg. Chemistry. (*To be continued.*)

Astronomy. — *"On the parallelism between radial velocity and intensity of light."* By Prof. W. DE SITTER.

(Communicated at the meeting of May 3, 1924).

In *Zeitschrift für Physik* XXI, 6, p. 333, 1924 M. LA ROSA points out that, if the velocity of a source of light is added to the velocity of light, then a star periodically approaching and receding from the observer will appear variable, as the quantity of light emitted by the star in equal intervals of time is perceived by the observer in intervals of unequal duration, the difference increasing with the distance of the star. This is, of course, entirely correct, but contrary to the opinion of Mr. LA ROSA, it does not afford an argument in favour of RITZ's theory of the propagation of light, but rather against it.

If the waves emitted by the source during the interval of time Δt reach the observer during the interval $\Delta t' = \Delta t(1+q)$, then the observer will ascribe to the source the intensity $i_0/(1+q)$, if i_0 be its real intensity, and on the other hand he will ascribe to it, according to DOPPLER's principle, a velocity of recession v determined by [1] $(c+v)/c = 1+q$, c being the velocity of light from a source which has no radial motion relatively to the observer. Both effects depend on *the same* factor q. We have thus, neglecting the square of q:

$$\frac{\Delta i}{i_0} = q = \frac{v}{c},$$

or since one stellar magnitude corresponds to a change of 0.4 in the common logarithm of the intensity

$$v = 277000\,\Delta m,$$

the difference Δm being expressed in stellar magnitudes, and the velocity in km. sec.$^{-1}$. Thus, if this were the real and the only explanation of the variability of stars, the change of observed wavelength, corresponding to a change of intensity of some tenths of a magnitude, would already be so large as to be interpreted as a velocity of the same order as the velocity of light itself. Inversely the change of magnitude, which undoubtedly accompanies any real change of velocity, is, for the velocities actually observed amongst stars, so small as to be entirely unobservable. To a change of velocity of 300 km. sec.$^{-1}$, which is about the largest velocity occurring amongst double stars, would correspond a change of 0.001 mag.

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[1] According to the classical theory, if the observer is at rest and the source is moving. According to the theory of relativity the formula of course is $\sqrt{(c+v)/(c-v)} = 1+q$.

Take as an example a star moving with uniform angular velocity n in a circle of radius a, of which the plane passes through the observer.

The distance from the star to the observer is

$$\Delta = \Delta_0 - a \sin nt,$$

and the component of the velocity of the star towards the observer is

$$v = an \cos nt.$$

If now the velocity of the light emitted by the star were

$$c' = c + \varkappa v,$$

where $\varkappa = 1$ for Ritz's theory and $\varkappa = 0$ for the ordinary theory, then the light leaving the star at the time t will reach the observer at the time

$$t' = t + \frac{1}{c}(\Delta_0 - a \sin nt)\left(1 + \varkappa \frac{an}{c} \cos nt\right)^{-1},$$

and consequently, if we neglect the squares and higher powers of an/c,

$$\frac{dt'}{dt} = 1 - \frac{an}{c} \cos nt + \varkappa \frac{an^2 \Delta}{c^2} \sin nt, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or

$$\Delta t' = \Delta t\,[1 - K \cos (nt + \varepsilon)], \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where

$$\tan \varepsilon = \varkappa \frac{n \Delta}{c}$$

$$K = \frac{an}{c} \sec \varepsilon.$$

La Rosa has neglected the first term in (1) and only taken account of the second term. Zurhellen (*A. N.* **198**, 4927, p. 1, 1914) has pointed out that the angle ε occuring in (2) would, if it reached at all appreciable values, in the case of an eclipsing binary give rise to a difference between the phase as derived from the observation of the eclipse and as derived from the radial velocity, and has concluded from the discussion of 7 stars that the value of $\varkappa$ must be smaller than one millionth.

Palaeontology. — "*Apateodus Corneti* (*For.*) *in the Senonian beds of the Southern part of Limburg* (*Netherlands*)." By P. KRUIZINGA. (Communicated by Prof. G. A. F. MOLENGRAAFF).

(Communicated at the meeting of December 29, 1923).

Some time ago an extremely well-preserved skull of a fossil teleostean, which was in possession of Mr. HUB. LEUFKENS[1]) of Heerlen, was found near Valkenburg in a small quarry in the chalk with *Belemnitella mucronata* Schloth.

In this specimen (fig. 1—4) only a few portions of the opercles and of the cheek plates have during the decay shifted from their normal position, whereby some have got lost. During the fossilization the skull was not flattened, but slightly pressed obliquely towards the right, so that all the bones are still connected in their normal relative position. While the skull was being extracted from the rock the whole was only slightly damaged.

First of all the matrix was prepared away on all sides, wherever it was possible to do so without interfering with the entire fossil. Thus on the left the nasal cavity and the orbit were cleaned and because the hyomandibular had been damaged a little on that side, this bone was also cleaned on the right side. Ipsilaterally the portions of the opercles were laid bare. Where it was possible, the teeth on either side were exposed. This could only imperfectly be done on the right side, because it was desirable to spare the remains of the premaxillary preserved on that side. At the lower surface the hyoid arch was carefully prepared out from the matrix, as was also the posterior part of the right half of the occiput. In doing so three vertebrae became visible in situ. Scales were not distinguished.

At first glance this skull from Valkenburg somewhat resembles that of a pike, especially when viewed from above; here also the snout is rather obtuse, but it is relatively shorter. Postero-anteriorly the width diminishes gradually. A lateral view of the skull shows that it is slightly flattened. Longitudinally the upperside of the skull is quite straight, transversely the cranial roof is flat between the eyes; behind, as well as before the eyes it is gently rounded towards either side. The occiput is concave, because the pterotics project further backwards than the epiotics and the supra-occipital. It is almost vertical to the cranial roof.

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[1]) When this communication was already in the press we obtained the skull from Mr. LEUFKENS for the collection of the Geological Museum at Delft, where it is now placed. (N⁰. 1273 Ned.).

The full length from the end of the snout to the posterior extremity of the existing remains of the opercula is 19.2 cm. It may be observed here, that of the opercula only small portions of a width of a few millimeters are lacking. The length of the upper side of the skull is 13.2 cm. The length of the mandible is 13.3 cm. Its suspensorium is, therefore, located just beneath the occiput, so that the mouth, which displays a large number of minute and large teeth, is very wide. The vertical extent from the top of the supra-occipital to the underside of the mandible is 9.4 cm. on the left, and 8.6 cm. on the right side. The height of the skull is in the middle at the back 3 cm. The width is at the back 5 cm., in the middle over the eyes 4.1 cm. and at 1 cm. from the snout it still attains 2.2 cm.

The orbit is almost circular and open towards the back, because the set of circumorbitals has almost entirely disappeared here. The hinder border is marked by a downward prominence of the sphenotic. The anterior border is removed 5.6 cm. from the snout, so that the orbit is located about half way between the snout and the occiput. The height is 4.1 cm., the length is about 4.5 cm. and the depth about 2 cm.

Before the orbit we still distinguish a smaller oblong pit, viz. the nasal cavity measuring 2.6 × 10 × 7 mm.

At this skull the following bones were observed:

supraoccipital,
pleurooccipital,
basioccipital,
epiotic,
pterotic,
sphenotic,
prootic,
alisphenoid,
basisphenoid,
parasphenoid,
orbitosphenoid,
pleuroethmoid,
mesethmoid with nasalia,
frontal, (preserved partly as an impression)
vomer,
palatine, (left and right, with one fang)
ectopterygoid, (left and right with one fang)
entopterygoid,

metapterygoid,
hyomandibular,
quadrate,
symplectic, (only on the left)
maxillary, (chiefly as an impression, the left one is very fragmentary)
premaxillary, (the left one with three minute teeth and traces of two, the right one a single toothlet)
dentary, (partly as an impression, the left one with six teeth, the impression of one tooth and traces of a couple, the right one with eight teeth and traces of a few others)
articulary,
angular,
stylohyal, (only the left one is visible)
epihyal,
ceratohyal,
basihyal,
preoperculum, (only the left one)
operculum, (only the left one)
suboperculum,
interoperculum,
branchiostegals, (only the right ones)
postorbitals?

At the back of the occiput there are still the first three vertebrae. Of the bones arranged in pairs both specimens are present, unless otherwise stated.

Of this species large and small fragments have repeatedly been detected, as is apparent from those occurring in several collections in our country and also elsewhere [1]; the detached teeth have in this connection been left out of consideration. We shall now give a short description of such remains as are in any way generally important. I deemed it desirable however, to add a complete picture of the anatomy of a skull of this species by utilizing all the collected data.

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[1] I gratefully acknowledge my indebtedness to Prof. ESCHER of Leyden, Prof. DOLLO of Brussels and Prof. FRAIPONT of Liege, Mr. LEUFKENS of Heerlen and Mr. UMBGROVE, Bachelor of Geology, of Leyden for granting me access to the above-mentioned remains, so that my inquiry could be as extensive as possible. I have much pleasure also in thanking Miss POPTA for allowing me the free use of skeletons of recent fishes in the "Museum van Natuurlijke Historie" of Leyden, and Prof. VAN BEMMELEN, Prof. VAN KAMPEN, Dr. TATE REGAN and Dr. DE BEAUFORT who furnished me with some books and publications that were not instantly accessible.

A fragment of a mandible (fig. 9) from the Upper-Senonian of the Southern part of Limburg that cannot be localized further, is in the Geol. Museum of Delft N°. 1264 Ned. It is an inner aspect of the anterior portion of the right jaw. Its length is 12.2 cm. The dentary, the only portion left, bears a large number of teeth, which will be described in detail on page 307.

At a shattered skull, from the Upper-Senonian of Valkenburg, and present in the Geol. Museum of Leyden (N°. 8594 of the STARING-collection) the following bones can be distinguished: both frontals, right pleuroethmoid, right ectopterygoid with one large tooth 2.5 cm. long, right palatine with an impression of the fang, mesethmoid, vomer, fragments of the right praemaxillary with still a single toothlet, right and left dentary, right and left articulary, left praeoperculum, rather well-preserved and a few not identifiable bone-fragments. The fragment is about 15 cm. long and 11 cm. broad.

A fragment of a skull, a gaping, though compressed mouth (fig. 10) found in a quarry in the Upper-Senonian at the road from Valkenburg to Sibbe, was lent me by Mr. UMBGROVE. Its length is 10 c.m, its height 5 cm. First of all we distinguish here: fragments of the right and left ectopterygoid with a few teeth. The posterior portion of the left ectopterygoid is beautifully preserved. In this bone 4 teeth are found that will be fully discussed lower down on page 306. In front of it we observe the apex of a larger one; the right ectopterygoid exhibits only a portion of a large tooth corresponding to the last-mentioned tooth in the left ectopterygoid. Of the mandible portions of left and right dentary are extant, the anterior portions are lacking. The left dentary is very solid. It displays the hinder border of this bone and contains still 4 teeth of various sizes. Of a couple of other teeth the apex has broken off. The right dentary is seriously damaged and partially preserved as an impression; it only contains the 3 teeth that correspond to the three front-teeth of the left dentary. The left articulary displays a large portion of the joint for the mandible, the right one is merely visible as an impression. Of the hyoid arch epihyal, ceratohyal and perhaps still remains of the anterior bones have been preserved.

Two of the fragments preserved in the Musée Royal d'Histoire Naturelle of Brussels will be discussed now in more detail, first of all a fragment of a skull from the Upper-Senonian of Kunrade N°. I G. 6921, belonging to the UBAGHS collection. This fragment is of special interest because the whole occiput and the major part of the sidewall of the skull have been very well preserved, while at the same time the sutures between the different bones are clearly visible (fig. 12 and 13). The width at the occiput is 3.7 cm. We see here the supraoccipital, both pleurooccipitals with the foramen magnum, the basioccipital, both epiotics, both pterotics and the two overlapped posttemporal grooves. At the left side of this skull-fragment we see also of the other cranial bones the prootic, the sphenotic and a portion of the parasphenoid. As regards the perforations the foramina in the prootic and in the pleurooccipital are conspicuous. At the dorsal side we finally observe fragments of the frontals. Here also the suture between the supraoccipital and the frontals is very distinct. The length of this piece is 5.5 cm. In the second place I mention a skull, originating from the BOSQUET-collection N°. I G. 4289 and found in the flintbearing tuff chalk of Valkenburg, of which the posterior portion has for the greater part disappeared (fig. 11). The length of the whole fragment is 13 cm. The mouth is slightly open, so that the matrix could be removed. The anterior portions of the frontals, the parasphenoid, the left pleuroethmoid, the mesethmoid, the vomer, the two palatines each with one fang were still present. The right palatine is partially damaged. We also distinguish the two ectopterygoidea with a few teeth and the left entopterygoid, besides fragments of the two praemaxillaries with 8 toothlets. The small fragment of bone lying in the orbit may be a portion of the right entopterygoid. In the mandible remains of the dentary are found on either side, on the left with some eleven teeth and on the right with ten, those in front being the smallest, and also the articulary. The greater part of the latter is merely an impression. Of several of the other bone-fragments some remains still exist here or there.

Lastly we have still to record the fragment, described by Forir as *Enchodus Corneti* (fig. 7 and 8). It is derived from the lowermost Maestricht Chalk of Valkenburg and is in the Geol. Museum of Liege. Unfortunately it was held upside down, so that the mandible has been described as an upper jaw. Consequently the figures and the description of this fossil have also to be altered. The back portion of the skull is quite absent and the snout is slightly damaged. The fragment is 7.3 cm. long, its breadth at the back is 3,8 cm., the height is 3.3 cm. The anterior rest of the parasphenoid, the vomer (chiefly as an impression), nasals (?), the two palatines are still recognizable. Of the fang of the right palatine only the impression is visible, that of the left palatine is present, but is slightly removed from the front. A fragment of the right ectopterygoid still contains a single large tooth of the length of 2.5 cm., that sticks out through the mandible; the left one has a similar tooth with two smaller ones behind it. The left praemaxillary has seven teeth while the impression of an eighth is noticeable; the right one has eight of them. Of the mandible parts of the two dentaries have been preserved: the left one with six teeth, three more are scattered among them having got loose from their original location; the right one has still five teeth, an additional detached tooth lying near. Of the left articulary probably the anterior portion is left. Finally a few fragments of the hyoid arch are to be seen, which probably are remains of the ceratohyals, the hypohyals and possibly of the basihyal.

In consideration of the size of the bone-fragments of all the remains described in this paper, we are bound to assume that all, except perhaps that of fig. 12, are derived from fishes of about the same size.

The less important remains, which I found in the Geol. Mus. of Leyden, were a symphysis of a mandible from St. Pietersberg, a fragment of a right half of a mandible found near Valkenburg and a few pieces of bone (ectopterygoid) with teeth from St. Pietersberg, and in the Museum at Brussels I saw two skull-fragments from Kunrade, a skull-fragment from Valkenburg and a portion of a mandible, found between Huls and Simpelveld. These latter fragments are like the preceding ones derived from the Upper Senonian and presumably from the lowermost stratum of the Maestricht chalk.

We shall now proceed to a more detailed description of a skull of uniform size to the first-named specimen.

A considerable portion of the supraoccipital constitutes part of the occiput, while another portion about equal in size belongs to the bones of the cranial roof. The first is a rather stout, more or less hexagonal, concave boneplate which attains a height of 1.2 cm. and a width of 1.6 cm. At the top the crista extends backwards as a rodshaped bone, 4 mm. in length. The other portion is slightly triangular and 1.3 cm. long. The sutures between the supraoccipital and the contiguous bones of the occiput are straight and well displayed as, indeed, are all the other sutures of the occiput. On the other hand those between the upper part of the occipital and the adjacent bones of the cranial roof are not so conspicuous.

The two pleurooccipitals enclose at the hinder part of the skull the foramen magnum and together with the basioccipital they also help to form the base of the cranium. The portion of the pleurooccipital that constitutes part of the occiput is apparently robust and also hexagonal. It attains a height of 1.3 cm. and its largest width is 1.6 cm. Right at the lower end of each there is a short process which overlies as a zygapophysis the body of the foremost vertebra. The other portion of these bones is about vertical to the first and is quadrangular, its length and height being 8 mm. Near the hinder margin it comprises the slightly oblong foramen of the nervus vagus (X). The foramen magnum is gently rounded, of triangular shape; its height is 5 mm., it is widest at the base (8 mm.).

The basioccipital fits to the first vertebra with an elliptical facet. Its height is 8 mm., its width 11 mm. It extends to a length of about 11 mm. along the base of the skull and is at its lower side provided with two keels that increase in width anteriorly.

On either side of the supraoccipital are the epiotics along the upper border of the occiput. They are more or less rhomboidal and stout bones. They do not form, as is often the case, distinct processes, but are slightly curved outwards in some places. The height is 8 or 9 mm., the width 10 mm. Their postero-superior aspect is triangular, measuring 5 mm. by 10 mm. It is these bones that Hay mistakes *in Empo nepaeolica* Cope for parietals (9 page 85 fig. 69).

Opisthotics are evidently lacking.

The pterotics are located at the upper side of the cranium, anterior to the epiotics and lateral to the frontals and the supraoccipital. To a smaller extent, however, they also go to form the lower side of the cranium and the occiput. These three parts are all approximately quadrangular. The pterotics make up the two exterior processes of the occiput, which, though not being very long, stretch farther backwards than the epiotics and the crista of the supraoccipital. The superior aspect of the occiput is, therefore, slightly concave. Moreover the upper part of the pterotics has roofed the posttemporal grooves, which are found on the left and on the right between epiotic and pterotic. Together with the contiguous sphenotic the lowermost piece of the pterotic supports a facet for articulation for the hyomandibular. The upper part of the pterotics is 2 cm. by 2.1 cm., the lower part is 8 cm. wide. The piece forming a part of the occiput is 8 mm. by 6 mm., and borders on the pleurooccipital and on the epiotic.

The sphenotics are very irregularly shaped bones, flanking the frontals and obliquely superior to the orbit. Posteriorly they border on the pterotics. Their shape somewhat resembles that of a trilateral pyramid, as they curve round the side of the cranium as well as round the rim of the orbit. The apex marks the hinder border of the orbit. Thus in front they are part of the wall of the orbit; as we saw before, on either part a portion forms together with the pterotic a facet for articulation for the hyomandibular. The length along the rim of the orbit is 1.6 cm., along the side of the cranium 1.8 cm., and along the wall of the orbit they are produced 1.2 cm.

The prootics are lying in the lateral walls of the cranium. Superiorly they border on the sphenotics and the pterotics, posteriorly on the pleurooccipitals and inferiorly on the wing-shaped appendices of the parasphenoid. They are pentagonal or hexagonal bones, 1.2 cm. in height. The width is only little greater. They are pierced by a few conspicuous foramina, one for the nervus trigeminus (V), a second for the nervus facialis (VII), while the others transmit various bloodvessels.

The parasphenoid is very long. The hinder part is contiguous to the basioccipital at rather more than 1 cm. from the occiput; anteriorly it still overlaps the hinder part of the vomer. Its length is certainly about 10 cm. At the hinder part the parasphenoid is provided with a couple of wings which, as we observed before, are connected with the prootics. The height is here 1,1 cm., which diminishes anteriorly rather rapidly, but before the wings disappear altogether we still find on either side of the parasphenoid a narrow rodshaped bone, which is at the top attached to the prootic. This leaves on either side a rather wide opening between the wing of the parasphenoid, the prootic and the rodshaped bone. The opening between these bones of the left and the right side is 1,2 cm. in height and 1 cm. in width. Where the wings end, the aspect of the parasphenoid is a transverse section of triangular form (height 3—4 mm.), and right in front it terminates as a thin, gutter-shaped, excavated bone.

The Y-shaped basisphenoid is lying a little in advance of the prootics on the middle of the parasphenoid, and in front of the opening just described (myodoma) at the very spot where the wings of this bone have disappeared. Its inferior part is constituted by a boneplate, measuring 1.4 cm. in length and 0.4 cm. in width. It is connected to the alisphenoid and the prootics by means of the two upper prongs which are much shorter. The total length of the basisphenoid is 1.8 cm.

The alisphenoids are of a very irregular shape. At the base they are very narrow, but further upwards they largely increase in width, inclining outwards, so that they help to form the roof of the orbit. The greatest length is ± 2 cm., the width 1 cm. Their hinder part borders on the prootics, their upper side on the sphenotics, while between the right and the left alisphenoid and the prongs of the basisphenoid we can still distinguish the opening for the forwardmoving blood-vessels, the nervus olfactorius (I) and the nervus ophthalmicus (II). Anteriorly the alisphenoids border on thin, plateshaped bones, which form part of the roof of the two orbits and proceed further as a united, unpaired orbitosphenoid, which descends as a semicircular plate in the non-ossified interorbital septum. The height of the vertical plate is about 1 cm., its length 1.7 cm.

At first the pleuroethmoids seemed to be short dumb-bell-shaped bones, 1.2 cm. in length and with a median thickness of 6 mm. They are situated anteriorly to each of the orbit between frontal and the arch of the palatine and pterygoids. On further preparation they proved to extend further inward into the frontwall of the orbit and into the hinderwall of the nasal cavity, as far as the parasphenoid, so as to approach each other from both sides. The shape and the surface of these inwardlyinclined wings are very irregular, their width is 1.3 cm.

The mesethmoid is a small bone (± 2 cm. in length) tapering anteriorly, and hemming in the V-shaped anterior end of the frontals. On either side it passes almost imperceptibly into an oblong bone, which overlies in front the palatine and is continued backward as far as the nasal cavity. This bone I, therefore, look upon as the nasal. Its length is 17 cm., its width 0.7 cm.

At the dorsal side of the cranium the frontals are by far the largest bones. The longitudinal suture between them is very conspicuous. Posteriorly they adjoin the supraoccipital and in front they come very near to the snout. So they are not separated from the supraoccipital by the parietals, as A. SMITH WOODWARD maintains for very closely related species (6 pl. 54 fig. 4*a* p. 246). He is presumably also wrong in representing the condition as on 6 pl. 11, fig. 1*b* and 8 page 216, since from this it would follow that the parietals are separated from each other through the contiguity of the frontals and the supraoccipital. So far as I can judge from the material at my disposal the parietals are lacking. The frontals attain a length of 10.4 cm. They attain their greatest width just behind the eyes where they are together 4 cm. in width. From this point backward they decrease in width rapidly though not very regularly. Anteriorly the width decreases only slowly. They hang over the eyes like a sheltering roof. On the surface there are a great many striae, which are still more conspicuous on the interior of the cranium than on the outward surface and which can be observed also in related species. All these striae radiate to all sides from the centra of ossification postero-superiorly to the eyes, thus forming in the middle of the head a large number of more or less rhombic figures. All other ornamentations are lacking.

The vomer is a much elongated, lancetshaped bone, with a longitudinal groove in the middle. In front of the snout it is visible under the frontals; its hinder part still is covered by the anterior portion of the parasphenoid. Its width is 4 mm., its length probably more than 4 cm.

The visceral skeleton articulates on either side of the cranium by a stout hyomandibular with the pterotic and the sphenotic. The surface of articulation is 2.8 cm. long and 4 mm. wide. Inferiorly the hyomandibular is first narrowing a little, but it soon gets broader again, the anterior margin being all along regularly rounded, while the hinder margin is cut obliquely. Downward it splits up into three branches, the hindmost being the first to appear. This branch is slightly inclined towards the inside; it is small, thin and probably connected the hyomandibular with the opercular. The median branch is short and tapers inferiorly. Its posterior side is straight and was originally lying along the anterior margin of the preoperculum. Between this median branch, the anterior one, the preoperculum and the symplectic there is a wide open space. The anterior branch is the largest. It is slightly bent and suturally united with the metapterygoid. Antero-superiorly to the hyomandibular a low crista is seen to begin, which is curving towards the open space just now alluded to.

The metapterygoid is a thin bone plate, 2.8 cm. by 1.7 cm., and is provided with a vertical ridge at the concave upper margin. The angles are considerably rounded off. Beneath this bone and also adjoining the hyomandibular the quadrate is observed. It is a rather stout bone, more strengthened still in some places; its general form is more or less triangular. The hinder margin is slightly round and displays a curve, just fit for the symplectic to fill up. The upper and the anterior margins are concave. On the lower end we see the condyle for the mandible. The length of the quadrate is 2.5 cm.; the greatest width 2.4 cm. The symplectic running along the hinder margin of a quadrate in the just-mentioned curve, is probably a rodshaped bone, of which we cannot determine the dimensions, since part of the only specimen that came to our notice has underlapped the quadrate.

Of the pterygoids the ectopterygoid is the principal. It extends from before the snout to the quadrate at the back, so that its length is 9 or 10 cm. A side-view shows that it is bent somewhat sigmoidally and antero-posteriorly it first broadens towards the anterior margin of the orbit, below the orbit it is narrow, but afterwards it broadens again. Quite at the back it is about 2 cm. in height. Anteriorly the ectopterygoid is getting rather slender towards the end. Along the upper rim of the anterior part of the ectopterygoid a plate is running on the inside bearing a number of stout teeth. Its thickness is not considerable, only ± 1 mm., antero-posteriorly it gets narrower and disappears in the middle below the orbit.

The entopterygoid resembles in shape a triangular, bellying sail. Its length is 4.2 cm., its width, which is greatest at the back, is 1.8 cm. It extends from the metapterygoid to close to the pleuroethmoid. The lower margin adjoins completely the superior and internal margin of the ectopterygoid. The upper margin runs up to 6 mm. from the parasphenoid.

It is very likely that the palatine stretches as far as the pleuroethmoid. It measures 0.7 cm. by 4 or 5 cm. A fang is implanted in front. This part of the bone is stout and slightly tumid, antero-posteriorly it passes into a thin, porous boneplate, which is provided with a number of longitudinal tubes ± 1 mm. in diameter. The foremost portion of the ectopterygoid is covered by this plate.

Of the bones in the upper jaw the premaxillary is the smallest. It is a thin more or less elongated boneplate, leaning on the palatine and the ectopterygoid. The upper rim is rounded, the lower one, on which a number of teeth are implanted, is slightly concave. From a little behind the anterior part of the palatine it extends to a little beyond the pleuroethmoid. The maxillary on the other hand is drawn out to a great length, rather narrow and thin. It has probably attained a length of 10 cm. and certainly not wider than 5 or 6 mm. No remains of a supramaxillary have been found. Only a few indications have been found of a series of circumorbitals, such as are described by A. Smith Woodward (p. 246, pl. 54) of *Apateodus striatus* A. S. Woodw., an allied species from the English Chalk. These indications consist in small bone-fragments in the orbit.

The mandible. In each branch of the mandible there are three bones, viz. dentary, articulary, and angular. No indications have been found that there should have been more of them originally. Its length is ± 13.3 cm. The shape is typical. Both halves display only a slight curvature in their long axis and they run nearly parallel over a long distance. Superiorly downward they are, moreover, slightly rounded. The hinder margin is vertical, the superior angle is rounded off. A little before the posterior end the height is greatest, viz. 2.8 cm., or $^1/_5$ of the length. Anteriorly it is constant all along the posterior half of the jaw, but then it decreases evenly, a little more rapidly at the symphysis. At the back we see two processes. The lowermost, which is the smallest, forms the infero-posterior angle; a little above it is the condyle.

The dentary. This bone is by far the largest of the bones in the mandible. On the whole it is not thick, especially the inferior portion. The upper border is sharp, but it is slightly thickened in the anterior half where the teeth are situated. It extends backwards over a considerable distance, as it forms the whole of the upper border and of the lower border only ± 2 cm. remains free at the posterior end. The posterior border of the dentary runs obliquely forwards from the lower border, then it bends round sharply and ultimately proceeds posteriorly in horizontal direction.

The articulary is a rather elongated, triangular bone. Its length is 7.6 cm. Its width behind is 2.3 cm. Infero-posteriorly we see the long process, ± 8 cm. in length, with the articulating surface. This portion of the bone is slightly thickened, while also a thickened strip extends forwards from the condyle. As appears from the above, only a small portion of the articulary is not covered by the dentary, viz. a more or less quadrangular piece of ± 1½ by 2 cm. The remainder wedges itself behind the dentary.

The angular, the smallest of the bones of the mandible, is only ± 1 cm. long and 3 mm. wide. It forms the inferior angle and sticks out backwards as a process of about 2 mm. in length.

The hyoid arch.

The stylohyals are small rodshaped bones, a little thicker at the lower side than at the upper. Its length is ± 8 mm., the greatest thickness ± 3 mm. The epihyals are a pair of flat bones broadening anteriorly. The greatest width is ± 1½ cm., the length 2½ cm., and the thickness ± 4 mm. The ceratohyals are rather long and bent slightly inwardly. Posteriorly they attain a height of 1.5 cm. At the extremities they are slightly thicker than in the middle viz. 4.5—5 mm. The length is 4.7 cm. The hypohyals are only 8.5 mm. in length, their thickness amounts to ± 5 mm. and their width is 1 cm. Of the foremost bone, the basihyal, only a horizontal section was observed. It is triangular in form, its width is ± 6 mm., its length about 4 mm.

Four of the branchiostegals have been preserved. They are thin, considerably elongated boneplates impinging on one of the epihyals and under the interoperculum and suboperculum still present there. The width cannot be observed, but it is certainly ± 1 cm. The length of the upper branchiostegal is 4.5—5 cm.

The opercular apparatus is complete, four opercles on each side. The preoperculum is approximately level, only along the anterior border the superior part is bent slightly outward. It is a rather large boneplate, approximately semicircular in circumference. The upper part of the anterior border, which was connected to the hyomandibular, is straight, the remaining part is curved. Most likely the height was 7 cm., the width ± 3.5 cm. The centre of growth is located anteriorly in the lower part. Here the bone is also thickest and from here a small number of fine lines radiate out to all sides. For the rest the surface displays only faint, concentric lines. The shape of the opercle is not well definable, the best-preserved specimen being broken in different places. However, from the concentric lines of growth we may conclude that the circumference has been more or less oval; the anterior, and the upper borders were curved less than the other sides. The width is ± 3.5 or 4 cm., the length ± 5½ cm. The subjacent suboperculum is, as far as the upper and the anterior border are concerned, covered by the other bones. It seems, however, that this bone is more or less triangular with truncated or rounded angles. The length is ± 6 cm., the width 3 or 4 cm. Fine concentric lines of growth are present. The interoperculum adjoining the mandible has a concave border at its front, where the thin boneplate is slightly thickened. The shape is oblong, the angles are rounded off. The width is 1½ cm., the length 3 cm. On the surface there are a number of fine striae to be distinguished, which radiate from a point very near to the anterior border.

As stated above, the teeth are situated at the palatines, the ectopterygoids, the premaxillaries and the dentaries. They not only vary in shape and size according to the various bones on which they are implanted, but also the teeth of one and the same bone are very different. Where teeth are implanted, the bone is slightly tumid. The palatine bears only one large fang, which may be subterminal, or at ± 1 cm. from the anterior end, a difference that may occur at one and the same skull, and that, to my judgment, is due to the teeth not being shed contemporaneously. The apex is turned to the inside. The tooth has fused with the bone, its base being thereby broadened, it is bent backward like a sickle and tapers to a sharp point. To the right and to the left it is flattened, at the base less than towards the apex. The transverse section is consequently rather oval-shaped at the base; towards the top the anterior and the posterior edge soon gets sharp. The upper surface is glistening and provided with delicate longitudinal striae, which are most conspicuous at the base. Where they are sharp, the anterior and the hinder edge are on either side provided with fine, transverse grooves, by which they are finely serrated. At the base the dimensions of the transverse section are ± 3 and 4 mm. The length of such a tooth is ± 1,1 cm.

The ectopterygoid bears a small number of teeth, which have grown together with this bone with a broadened base. Before the pleuroethmoid there may perhaps be two, rather widely spaced. (See also 6, pl. 11, fig. 8, fig. 3, and fig. 4. The portion under consideration, however, is designated there as palatine). The foremost of these teeth was not present in any of the specimens at my disposition. The other is located obliquely below the pleuroethmoid. Its length is about 2.5 cm., its width 7 mm. In the fragment originally described as *Enchodus Corneti* For. its point sticks through the lower jaw. The shape is dagger-like. At the base the transverse section is again oval, and towards the tip the tooth is flattening more and more on both sides. The anterior, and the hinder edges are both sharp and provided with fine transverse grooves. For the rest only longitudinal striae are to be seen on the glistening surface, which are most distinct at the base just as with the fang of the palatine. The other teeth of the ectopterygoid are all smaller, their size decreasing towards the end of the series. On the whole, however, all the teeth of this bone agree fairly well as to the shape. The spaces between them differ according as old teeth have or have not been dropped and replaced by new ones, so that also the number may be different. All of them have their apex turned slightly inwards.

Dimensions of the teeth of the ectopterygoid (without the base) of the fragments represented in fig. 7 and 10.

	length mm.	width mm.
1st tooth (fig. 10)	25	7
2nd tooth (fig. 7)	18	6,5
3rd tooth	9	4,5
4th tooth	9	4
5th tooth	6	3,5

The premaxillary bears only some very small teeth. I counted 8 at most (type spec. of *Enchodus Corneti* For., on the right premaxillary). They are implanted on the lower border of this bone at various distances from each other and are more turned inwards than the preceding. On either side they are slightly flattened, so that their anterior and their hinder edges are sharp. The surface again presents fine, longitudinal striae. The length ranges from 1½ to 2mm. and at the base the width is ± 1 mm.

The dentary bears the greatest number of teeth of largely differing sizes. They form one series on a thickened ridge that runs internally just below the upper border, as is also the case with the teeth of the ectopterygoid. In front near the symphysis we invariably find some very minute teeth, about five in number. Behind them we observe a tooth that is a little larger and then follow the two largest teeth of the mandible; the teeth behind them are smaller again. Their size, however, does not decrease regularly; the hindmost, however, are very small. Their distances are varying for the reason mentioned before, so that also their number may vary. Only the anterior half of the dentary bears teeth. The mandible illustrated in fig. 9 presents more teeth than any other represented.

Counting backwards these teeth are of the following dimensions:

	length (mm.)	width (mm.)
1st tooth	1	0.5
2nd tooth	1	0.5
3rd tooth	1	0.5
4th tooth	1	1
5th tooth	4.5	2
6th tooth	10.5	4
7th tooth	12	5
8th tooth	5 to 6	3
9th tooth	5	3
10th tooth	8	4
11th tooth	7	4
12th tooth	1	1

So far as it can be ascertained all the teeth in the dentary are slightly curved backward. The inner, and outer faces are flattened, the outer face most. The diameter at the base is oval. The anterior edge above the base is sharp; the hinder edge on the other hand is rounded, except at the tip, where it also becomes sharp. The transition is marked by a slight notch in the hinder edge, which imparts to these teeth a recurved appearance, that we could also distinguish at the teeth of the ectopterygoid of *Cimolichthys*. Whether this feature belongs to the small teeth in the mandible as well as to the large ones, was not ascertainable. At the anterior edge we observe again the fine, transverse grooves. We see them also at the sharp portion of the hinder edge. The teeth of the mandible again are ornamented with fine, longitudinal striae.

Of the vertebrae only the first three are known for certain. The foremost is not so long as the two following, its length being only 4 mm., whereas that of the second is 6 mm., and of the third 7 mm. The diameter is invariably about 1 cm. There is, moreover, another difference: along the circumference of the foremost vertebra there are only two depressions, separated from one another by a broad strip. The other two have three deeper depressions, separated by rather narrow horizontal ridges. The vertebrae are amphicelous. The neurapophyses are ± 4 mm. in width. Their length was presumably ± 1½—2 cm. Apparently the neurapophyses were not coalesced with the vertebrae, from which they are evidently separated by a little of the rock. Such a residue is always found whereever bones are united by tissues that were originally softer (see e.g. the hyoid arch). The zygapophyses are small and were partially broken off while preparing them out.

As observed before, Forir (3) named one of the fragments described in this paper *Enchodus Corneti.*

When we consider the characteristics of the genus *Enchodus* as they have been described by Agassiz (1 page 64) and by A. Smith Woodward (8 page 190 and 6 page 55), Forir's pronouncement appears to be erroneous. For the sake of comparison I quote the following important passage from the work of the latter author:

"Trunk elongate-fusiform, both this and the head laterally com-"pressed. Cranial roof exhibiting a deep median longitudinal de-"pression its lateral and occipital margins ornamented, like the other "external bones, with ridges and tubercles of ganoine. Mandible a "little prominent, provided with an inner widely-spaced series of "large slender teeth, the largest in front, also a marginal series of "minute teeth, all nearly or completely solid; premaxilla in the "form of a vertical lamina, deepest in front, tapering behind, and "with a single spaced series of small teeth; maxilla long and slender, "either finely toothed or toothless at the oral border; palatine thickened "and tumid, with only one large tooth fixed at its anterior end; "ectopterygoid robust, with a single spaced series of large slender "teeth, gradually diminishing in size backwards; no teeth barbed. "Preoperculum very narrow and deep; operculum strengthened on "the inner side by a ridge extending horizontally backwards from "the point of suspension; branchiostegal rays about 12 to 16 in "number".

Now, none of the bones of the cranium of the species here described, is ornamented with tubercles of ganoine, while the mandible has only one series of teeth, the smallest in front, the largest behind them and the preoperculum is broad. Furthermore, the vertebrae of *Enchodus* appear to have only two depressions on each side (10, p. 184), while, with the exception of the foremost, the others have three grooves. However, the species agree in that also *Enchodus* has decidedly barbed teeth; only they are not so well developed. The mandible of *Enchodus Faujasi* Ag. bears teeth, more or less S-shaped and flattened laterally. All along the anterior edge these teeth are sharp; of the posterior edge only the upper third is sharp. Transverse grooves were not observed along these sharp edges.

A. Smith Woodward has referred the fragments of the species here described, which are in the Museum of Brussels, to the genus *Apateodus* (6, p. 38), as also appears from the accompanying labels written by himself. The report of the various features of this genus must, however, be brought up to date after the latest observations.

Generic characters of Apateodus. Skull triangular, the cheeks flat, superior aspect of the snout rather blunt, upper side of the skull straight, occiput a little concave and vertical, no special ornamentation at the bone beyond the lines on the frontals, which are radiating out from points above the eyes, point of suspension of the mandible beneath the occiput, the supraoccipital in contact with the frontals, the supraoccipital possesses only a very small crista, parietals are lacking, an orbitosphenoid, a basisphenoid and a myodome are all there, the palatine bears one fang, and overlaps part of the ectopterygoid, the ectopterygoid has one series of large dagger-like teeth, decreasing in size backwards, their anterior, and posterior edge is sharp and flattened on both sides, as is also the fang of the palatine, the premaxillary bears one series of toothlets and together with the maxillary, constitutes the borders of the upper jaw, height of the mandible one fifth of its length, hinder border vertical, superior angle rounded, inferior angle projecting, one series of teeth, in front small ones, then follow the largest, and then smaller ones again. The teeth of the mandible are also flattened on either side, but the inner face less than the outer face, and they are recurved, anterior edge sharp, hinder edge rounded except at the very tip, where the hinder edge is also sharp, so that it is slightly barbed, preoperculum broad, circumorbital ring present, branchiostegals long and rather broad boneplates, vertebrae provided with grooves, separated by ridges, the anterior one has broad ridges and two grooves, the other has narrower ridges and three grooves; the foremost vertebra is much shorter than the following.

Up to the present we know only three species of the genus *Apateodus* from the English Chalk, viz. *Apateodus glyphodus,* C. C. Blake from the Gault, *Apateodus striatus* (Ag.) A. S. Woodw., and *Apateodus lanceolatus* A. S. Woodw. from the Turonian.

Of *Apateodus glyphodus* C. C. Blake only a few jaw-fragments have been found. This species differs from the fragments, described in this paper, in that the series of minute teeth, which are exhibited in the mandible before the larger ones, occupy a much wider space than those in our species; their number also seems greater. From *Apateodus lanceolatus* A. S. Woodward our species differs in the position of the teeth in the ectopterygoid (part of which is called palatine by Woodward) which do not point forward, but are directed vertically downward or slightly backward. The anterior and the posterior edges of the teeth in *Apateodus lanceolatus* A. S. Woodward are straighter and the teeth are also more slender.

The differences between our species and *Apateodus striatus* (Ag.) A. S. Woodward are easiest to determine, since also of the latter the remains are much more complete. The two species are readily discriminated by the smaller angle at which we see the snout of *Apateodus striatus* (Ag.) A. S. Woodw. from a lateral aspect, as compared with our species; also by the relatively broader frontals at the upper side of the skull of *Apateodus striatus,* and by the shape of the part of the cranial roof that is made up of the supra-occipital and the hinder part of the frontals (6 pl. 54).

This leads me to conclude that our species differs from the three English species. I, therefore, propose to designate it by the name of *Apateodus Corneti.*

Originally A. Smith Woodward referred *Apateodus* to the family of the *Scopelidae* (8 page 258), but already soon afterwards he has classed this genus under the family of the *Enchodontidae* (6 page 37), a family he thinks to be related to the *Scopelidae,* the *Odontostomidae* and the *Alepidosauridae.* In the manuals of Boulenger (11 page 611) and of Goodrich (12 page 399), as well as in that of Zittel (16 page 132), we still find *Apateodus* placed in the family of the *Scopelidae,* while in the latest classification of fishes namely that by C. Tate Regan (13 page 131) we find *Apateodus* classified near the family of the *Alepidosauridae* in the order of the *Iniomi.*

Apateodus cannot belong to the *Scopelidae* because the premaxillary does not exclude altogether the maxillary from the border of the upper-jaw, and because the supraoccipital is contiguous to the frontals. Neither can it be connected with the *Alepidosauridae*, which do not possess an orbitosphenoid, while the structure of the skull of *Apateodus* does not admit of giving this species another place in the order of the *Iniomi*. I regret not to have found an opportunity to visit London in order to compare with the recent and with the fossil material. However, the properties of this material that have now become known, made me refer *Apateodus*, according to the classification of TATE REGAN, to the order of the *Isospondyli* as defined by him (13, page 77), notably to the suborder of the *Stomiatoidei*, a group of pelagian, physostomous fishes, some of which live at a great depth. Among them he includes e.g. the following families: the *Astronesthidae*, the *Chauliodontidae*, the *Stomiatidae*, and the *Malacosteidae*, which possess no or only very small parietals, and of the fossil-fishes the *Enchodontidae*, which as to structure of the skull bear a great resemblance to *Astronesthes* (17, page 613) and the *Stomiatidae* (13, page 82). After all, it seems to me, that *Apateodus* belongs to the family of the *Enchodontidae*. This view may be substantiated by a comparison of pictures of the different genera of that family with those of *Apateodus*, e.g. those of *Enchodus Faujasi* Ag. by SMITH WOODWARD (8, pl. 11 fig. 6) and of *Empo nepaeolica Cope* by HAY (9 page 85 fig. 69). But also when we try to refer *Apateodus* to the system of fishes after BOULENGER and GOODRICH, the result is, that we approach the family of the *Enchodontidae*.

In virtue of the foregoing I include *Apateodus* among this family. In accordance with TATE REGAN's opinion this family belongs to the suborder of the *Stomiatoidei* of the order of *Isospondyli* of the *Teleostei*. It must be concluded, therefore, that the genus *Apateodus* has been a genus of physostomous fishes of prey, which most likely inhabited the open sea. There is, however, no ground for assuming that the species belonging to it, were living at large depths. The facies of the rock both in Holland and in England go against such an assumption. At present the genus *Apateodus* is known to have been found from the Gault to the Upper Senonian.

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EXPLANATIONS OF THE FIGURES.

Fig. 1. Left lateral aspect of the skull with fang in the palatine, the pterygoids, basisphenoid, orbitosphenoid, symplectic, stylohyal, angular, pterotic, sphenotic, etc. 0.5 nat. size. Coll. Delft.

Fig. 2. Right lateral aspect of the skull with the premaxillary, remains of the maxillary, the dentary, the opercles and the condyle of the mandible. 0.5 nat. size.

Fig. 3. Upper side of the skull. Frontals impinge on the supraoccipital, parietals lacking. 0.5 nat. size.

Fig. 4. Underside of the skull with the hyoid arch, the branchiostegals and the angular. 0.5 nat. size.

Fig. 5. The first three vertebrae of fig. 2 at the basioccipital, about nat. size, part of the hyomandibular with the process to which the opercular was united.

Fig. 6. Left orbit of fig. 1, 0 8 nat. size, with orbitosphenoid, alisphenoid, basisphenoid, parasphenoid, prootic with foramina, myodome.

Fig. 7. Left side of skull-fragment. Upper-Senonian found near Valkenburg, 0.8 nat. size, origin. described as Enchodus Cornetic For. = Apateodus Corneti (For.) with palatine with fang, remains of premaxillary, ectopterygoid and dentary with teeth. Coll. Liege.

Fig. 8. Fragment of fig. 7, about nat. size, seen from above with vomer and the anterior end of the parasphenoid.

Fig. 9. Internal aspect of a fragment of a jaw, origin the Upper Senonian of Southern part of Limburg, teeth, 0.9 nat. size. Coll. Delft.

Fig. 10. Gaping mouth, found near Valkenburg, Upper Senonian, 1.2 nat. size, with the teeth in the ectopterygoid, and the hinder border of the dentary.

Fig. 11. Fragment of a skull Upper Senonian, found near Valkenburg, Coll. Bosquet in the Brussels Museum N°. I. G. 4289, 0.8 nat. size. Left lateral aspect of the skull with gape, the posterior end of the palatine, which overlaps part of the ectopterygoid, the fang in both palatines, the minute teeth in the premaxillary, the teeth in the left ectopterygoid and the teeth of the mandible (one series).

Fig. 12. Fragment of a skull. Upper-Senonian, found near Kunrade. Coll. Ubaghs in the Mus. of Brussels. N°. I. G. 6921. About nat. size. The bones of the occiput.

Fig. 13. Fragment of Fig. 12, nat. size, inferior aspect, part of the left inferior angle of the basioccipital wanting.

Fig. 14. Part of fig. 2 with the hyomandibular, metapterygoid and quadrate with articulation of the mandible, about nat. size.

LIST OF ABBREVIATIONS USED IN THE FIGURES.

s.o.	supraoccipital.	*f.*	frontal.	*an.*	angular.
p.o.	pleurooccipital.	*v.*	vomer.	*s.h.*	stylohyal.
b.o.	basioccipital.	*p.*	palatine.	*e.h.*	epihyal.
e.ot.	epiotic.	*ec.p.*	ectopterygoid.	*c.h.*	ceratohyal.
pt.ot.	pterotic.	*en.p.*	entopterygoid.	*h.h.*	hypohyal.
s.ot.	sphenotic.	*m.p.*	metapterygoid.	*b.h.*	basihyal.
p.ot.	prootic.	*h.*	hyomandibular.	*p.op.*	preoperculum.
a.s.	alisphenoid.	*q.*	quadrate.	*op.*	operculum.
b.s.	basisphenoid.	*s.*	symplectic.	*s.op.*	suboperculum.
p.s.	parasphenoid.	*m.*	maxillary.	*i.op.*	interoperculum.
o s.	orbitosphenoid.	*p.m.*	premaxillary.	*br.*	branchiostegals.
p.e.	pleuroethmoid.	*d.*	dentary.	*p.o.*	postorbitals.
m.e.	mesethmoid.	*ar.*	articulary.		

Mathematics. — *"On certain series of* Mr. HARDY". By Prof. J. C. KLUYVER.

(**Communicated at the meeting of March 29, 1924**).

In 1910 Dr. VACCA[1]) gave a curious expansion of EULER's constant C, another proof of which and some generalisations were given by Mr. GLAISHER[2]). In 1912 Mr. HARDY[3]) again treated Dr. VACCA's formula, thereby remarking that it is possible to obtain corresponding expansions for the coefficients C_h of the series

$$\zeta(s)=\frac{1}{s-1}+C+\sum_1^\infty C_h\,(s-1)^h\,.$$

As an example of these expansions Mr. HARDY deduced a series for C_1, but the expansion is rather complicated and no indication is given as to the nature of the expansions of the succeeding coefficients. In this note I will expand C_h in a more regular and general form. As in the formulae of Mr. HARDY, however, the series involve irrational terms and the convergence is too slow to admit a direct calculation of the constants C_h.

1. In the first place I consider a function $F(x)$ of the real variable x in general continuous and having a definite derivative $F'(x)$. If discontinuities exist, I assume them to occur only at integer values of x. In this case, for any integer n, I take A_n to be the value of $F(n)$. Finally I suppose that for large values of x (integer or not integer) the function $F(x)$ satisfies the conditions

$$|\,F(x)\,|=O\,x^\mu\,,$$
$$|\,F'(x)\,|=O\,x^\mu\,,$$

μ being a given positive number.

Then choosing a positive integer a as a base of logarithms and putting for any integer k

$$\beta_k=a-1,\ \text{if } k \text{ is a multiple } a,$$
$$\beta_k=\ \ -1,\ \text{if } k \text{ is not a multiple of } a.$$

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[1]) Quarterly Journal XLI, p. 363.

[2]) Ibid. XLI, p. 365.

[3]) Ibid. XLIII, p. 215.

I consider the series

$$S = \sum_{k=1}^{k=\infty} \frac{\beta_k}{k} F({}^a\!\log k) = \sum_{k=1}^{k=\infty} u_k\,,$$

and proceed to prove its convergence.

Putting

$$U_n = \sum_{k=an}^{k=an+a-1} u_k = \sum_{m=1}^{m=a-1} \left\{ \frac{F({}^a\!\log an)}{an} - \frac{F({}^a\log(an+m))}{an+m} \right\},$$

and supposing $F(x)$ everywhere continuous, from

$$\left| \frac{d}{dt}\left(\frac{F({}^a\log t)}{t} \right) \right| = O\,\frac{({}^a\log t)^\mu}{t^2}$$

we have at once

$$|U_n| = O\,\frac{({}^a\!\log n)^\mu}{n^2}.$$

If, however, $F(x)$ has a discontinuity for $n = a^p$, ${}^a\log n = p$, we have

$$|\,U_n\,| = O\frac{p^\mu}{a^p}.$$

Therefore in either case the series $\sum\limits_1^\infty |U_n|$ converges, and as $\lim\limits_{k=\infty} u_k = 0$ we may also conclude to the convergence of the series $\sum\limits_1^\infty u_k$.

In order to find a convenient expression for the sum S of the series, I consider a group of terms

$$G_q = \sum_{k=a^q}^{k=a^{q+1}-1} u_k = -\sum_{k=a^q}^{k=a^{q+1}-1} \frac{F({}^a\!\log k)}{k} + \sum_{k=a^{q-1}}^{k=a^q-1} \frac{F({}^a\!\log ak)}{k}.$$

Now, if we put

$$F(x+1) - F(x) = f(x),$$

and note that

$$G_0 = -\sum_{k=1}^{k=a-1} \frac{F({}^a\!\log k)}{k},$$

we get

$$S = \lim_{n=\infty} \sum_{q=0}^{q=n} G_q = \lim_{n=\infty} \left\{ -\sum_{k=a^n}^{k=a^{n+1}-1} \frac{F({}^a\!\log k)}{k} + \sum_{k=1}^{k=a^n-1} \frac{f({}^a\!\log k)}{k} \right\}.$$

But with the restrictions imposed on $F(x)$, we have evidently

$$\lim_{n=\infty} \left\{ \int_{a^n}^{a^{n+1}-1} \frac{F({}^a\log t)}{t} dt - \sum_{k=a^n}^{k=a^{n+1}-1} \frac{F({}^a\log k)}{k} \right\} = 0;$$

hence we may write

$$S = \lim_{n=\infty} \left\{ -\log a \int_n^{n+1} F(x)\, dx + \sum_{k=1}^{k={}^{an}-1} \frac{f({}^a\log k)}{k} \right\}.$$

So for instance putting $F(x) = 1$, $f(x) = 0$, we get

$$-\log a = \sum_{k=1}^{k=\infty} \frac{\beta_k}{k},$$

a simple result that is easily verified.

2. We now will make a definite choice for $F(x)$, and to this end I consider the function

$$g_h(x) = -2 \sum_{n=1}^{n=\infty} \frac{1}{(2\pi n)^h} \cos\left(2\pi nx - \frac{h\pi}{2}\right).$$

The function $g_h(x)$ is periodic and for $h > 1$ continuous. We have

$$g_h(x+1) = g_h(x) \quad , \quad g'_h(x) = g_{h-1}(x) \quad , \quad |g_h(x)| < \frac{2}{(2\pi)^h} \zeta(h).$$

Supposing $h = 1$, we have still in general

$$g_1(x+1) = g_1(x) \quad , \quad g'_1(x) = g_0(x) = 1 \quad , \quad |g_1(x)| \leq \tfrac{1}{2},$$

but $g_1(x)$ is discontinuous for integer values of x, and $g_1(n)$ is not defined. I will retain the periodicity for integer values of x, consequently I take

$$g_1(0) = g_1(1) = \ldots = g_1(n) = A.$$

Denoting the largest integer contained in the arbitrary number x by $[x]$, we have

$$g_1(x) = x - [x] - \tfrac{1}{2},$$

and again retaining this relation for integer values of x, we get

$$g_1(n) = A = n - [n] - \tfrac{1}{2}.$$

and thus we define the value $[n]$ of the discontinuous function $[x]$.

At the same time I consider the function $\bar{g}_h(x)$ defined by the formula

$$\frac{y\, e^{xy}}{e^y - 1} = 1 + \sum_{h=1}^{h=\infty} y^h\, \bar{g}_h(x).$$

It is seen that $\bar{g}_h(x)$ is a polynomial and that we have

$$\bar{g}_h(x) = \frac{x^h}{h!} - \tfrac{1}{2} \cdot \frac{x^{h-1}}{(h-1)!} + \frac{B_1}{2!} \cdot \frac{x^{h-2}}{(h-2)!} - \frac{B_2}{4!} \cdot \frac{x^{h-4}}{(h-4)!} + \frac{B_3}{6!} \cdot \frac{x^{h-6}}{(h-6)!} - \dots,$$

the final term being either $\dfrac{B_{\frac{h-1}{2}}}{(h-1)!} \cdot \dfrac{x}{1}$ or $\dfrac{B_{\frac{h}{2}}}{h!}$, and B_k standing for the Bernoullian number of index k.

Again the relation

$$\bar{g}'_h(x) = \bar{g}_{h-1}(x)$$

holds, but there is no periodicity and now we get

$$\bar{g}_h(x+1) = \bar{g}_h(x) + \frac{x^{h-1}}{(h-1)!}\ , \quad |\bar{g}_h(x)| = O\, x^h\, .$$

Although $g_h(x)$ and $\bar{g}_h(x)$ largely differ in nature, they are intimately connected. In fact, it is readily seen that

for $h > 1$: $g_h(x) = \bar{g}_h(x)$, if $0 \leq x \leq 1$,

for $h = 1$: $g_1(x) = \bar{g}_1(x)$, if $0 < x < 1$.

Hence we have in general

$$g_h(x) = \bar{g}_h(x - [x]),$$

but it is to be noted, that if we wish to retain this relation for integer values of the variable, necessarily we are lead to

$$g_h(n) = g_h(0) = \bar{g}_h(0) = \bar{g}_h(1) = \bar{g}_h(n - [x]),$$

that is either to $[n] = n$, $g_1(0) = A = -\frac{1}{2}$, or to $[n] = n-1$, $g_1(0) = A = +\frac{1}{2}$.

3. Evidently both functions $g_h(x)$ and $\bar{g}_h(x)$ fulfill the conditions imposed on $F(x)$, and taking $F(x) = g_h(x)$, we have for any integer h

$$f(x) = g_h(x+1) - g_h(x) = 0,$$

$$\int_{n}^{n+1} F(x)\,dx = g_{h+1}(n+1) - g_{h+1}(n) = 0.$$

Therefore we get as a first result

$$W_h = \sum_{k=1}^{k=\infty} \frac{\beta_k}{k}\, g_h\,({}^{a}log\, k) = 0,$$

the value of $g_1(0) = A$ being quite irrelevant if $h = 1$.

Generalising, we can take $F(x) = g_h(x - {}^{a}log\, b)$, b being any positive quantity, and in the same way as before we infer that also

$$W_{h,b} = \sum_{k=1}^{k=\infty} \frac{\beta_k}{k}\, g_h\left({}^{a}log \frac{k}{b}\right) = 0.$$

Again, putting

$$F(x) = a^{y g_1(x - {}^{a}log\, b)},$$

we find, whatever b may be, $f(x) = 0$ and

$$\int_{n}^{n+1} F(x)\,dx = \frac{1}{y \log a}\, a^{y g_1(x - {}^{a}log b)} \Bigg|_{x=n}^{x=n+1}.$$

Not overlooking the point of discontinuity in the interval of integration, it follows that

$$\int_{n}^{n+1} F(x)\,dx = \frac{Sh\left(\frac{y}{2} \log a\right)}{\frac{y}{2} \log a},$$

and thus we have the formula

$$-\frac{S_h\left(\frac{y}{2} \log a\right)}{\frac{y}{2}} = \sum_{k=1}^{k=\infty} \frac{\beta_k}{k}\, a^{y g_1\left({}^{a}log \frac{k}{b}\right)},$$

from which we conclude to the relations

$$\sum_{k=1}^{k=\infty} \frac{\beta_k}{k}\, g_1^{2n+1}\left({}^{a}log \frac{k}{b}\right) = 0,$$

$$\sum_{k=1}^{k=\infty} \frac{\beta_k}{k}\, g_1^{2n}\left({}^{a}log \frac{k}{b}\right) = -\frac{\log a}{2^{2n}\ 2n+1)}.$$

Next, we will take $F(x) = \overline{g}_h(x)$. In this supposition we have

$$f(x) = \overline{g}_h(x+1) - \overline{g}_h(x) = \frac{x^{h-1}}{(h_{-1})!},$$

$$\int_n^{n+1} F(x)\,dx = \overline{g}_{h+1}(n+1) - \overline{g}_{h+1}(n) = \frac{n^h}{h!}$$

and therefore, putting $a^n = m$,

$$\overline{W}_h = \sum_{k=1}^{k=\infty} \frac{\beta_k}{k}\,\overline{g}_h({}^{a}\log k) = \frac{1}{(\log a)^{h-1}} \cdot \frac{1}{(h-1)!} \operatorname*{Lim}_{m=\infty} \left\{ \sum_{k=1}^{k=m} \frac{(\log k)^{h-1}}{k} - \frac{(\log m)^h}{h} \right\}.$$

Now in the expansion

$$\zeta(s) = \frac{1}{s-1}\,C + \sum_1^{\infty} C_h\,(s-1)^h$$

the coefficients C_h are defined by the equation

$$C_h = \frac{(-1)^h}{h!} \operatorname*{Lim}_{m=\infty} \left\{ \sum_{k=1}^{k=m} \frac{(\log k)^h}{k} - \frac{(\log m)^{h+1}}{h+1} \right\};$$

hence we get the principal result of this note

$$\overline{W}_h = \sum_{k=1}^{k=\infty} \frac{\beta_k}{k}\,\overline{g}_h({}^{a}\log k) = \frac{C_{h-1}}{\left(\log \frac{1}{a}\right)^{h-1}}.$$

Thus then we have deduced convergent series, representing the constants C_h; and noticing the form of the polynomial $\overline{g}_h(x)$, it is seen that each constant C_h is equal to an aggregate of the series

$$Q_h = \frac{1}{h!} \sum_{k=1}^{k=\infty} \frac{\beta_k}{k}\,({}^{a}\log k)^h \quad , \quad Q_0 = -\log a.$$

So for instance we have

$$+\;C = Q_1 - \tfrac{1}{2}\,Q_0,$$

$$-\frac{C_1}{\log a} = Q_2 - \tfrac{1}{2}\,Q_1 + \frac{B_1}{2!}\,Q_0,$$

$$+\frac{C_2}{\log^2 a} = Q_3 - \tfrac{1}{2}\,Q_2 + \frac{B_1}{2!}\,Q_1,$$

$$-\frac{C_3}{\log^3 a} = Q_4 - \tfrac{1}{2}\,Q_3 + \frac{B_1}{2!}\,Q_2 - \frac{B_2}{4!}\,Q_0,$$

. . . .

the integer a being arbitrary, and taking $a=2$, we have simply $\beta_k=(-1)^k$. Inversely, we can express the series Q_h by the constants C_h. From the relation

$$\frac{({}^a\!\log k)^h}{h!}=\bar{g}_{h+1}(1+{}^a\!\log k)-\bar{g}_{h+1}({}^a\!\log k)$$

it readily follows that

$$Q_h=-\frac{\log a}{(h+1)!}+\frac{C}{h!}-\frac{C_1}{(h-1)!\log a}+$$
$$+\frac{C_2}{(h-2)!\log^2 a}-\frac{C_3}{(h-3)!\log^3 a}+\ldots+\frac{(-1)^{h-1}C_{h-1}}{1!\log^{h-1} a}.$$

We may remark here that the consideration of the analogous series

$$W_h^{(s)}=\sum_{k=1}^{k=\infty}\frac{\lambda_k}{k^s}g_h({}^a\!\log k),$$

$$\overline{W}_h^{(s)}=\sum_{k=1}^{h=\infty}\frac{\lambda^k}{k^s}\bar{g}_h({}^a\!\log k),$$

where λ_k stands for a^s-1 if k is a multiple of a and otherwise is equal to -1, does not lead to interesting results, as soon as $s>1$. For in the latter supposition the series $\sum_1^\infty \frac{1}{k^s}$ converges, and so it almost evident that we have

$$W_h^{(s)}=0,$$

$$\overline{W}_h^{(s)}=\frac{1}{\log^{h-1} a}\cdot\frac{(-1)^{h-1}}{(h-1)!}\,\frac{d^{h-1}}{ds^{h-1}}\zeta(s).$$

4. The convergence of the series Q_h is very slow and its terms are irrational. Only in the case $h=1$ a transformation into a series of rational terms is possible. In fact, for $h=1$ we have

$$C=\sum_{k=1}^{k=\infty}\frac{\beta_k}{k}\bar{g}_1({}^a\!\log k),$$

$$0=\sum_{k=1}^{k=\infty}\frac{\beta_k}{k}g_1({}^a\!\log k),$$

and as

$$\bar{g}_1(x)-g_1(x)=[x],$$

subtraction gives

$$C=\sum_{k=1}^{k=\infty}\frac{\beta_k}{k}[{}^a\!\log k],$$

a formula in which

$$[n] = n - A - \tfrac{1}{2},$$

where A has any value whatever.

If we take $[n] = n$, $A = -\frac{1}{2}$ we have the generalisation of Dr. Vacca's series as given by Mr. Glaisher, if we take moreover $a = 2$ we have Dr. Vacca's series itself

$$C = \sum_{k=1}^{k=\infty} \frac{(-1)^k}{k} \, [{}^2\!\log k].$$

We can modify this expansion in different ways. Since

$$\bar{g}_1\left({}^a\!\log \frac{k}{b}\right) = \bar{g}_1\,({}^a\!\log k) - {}^a\!\log b,$$

and therefore

$$C = \sum_{k=1}^{k=\infty} \frac{\beta_k}{k} \left\{ \bar{g}_1\left({}^a\!\log \frac{k}{b}\right) + {}^a\!\log b \right\},$$

we get by subtracting the equation

$$0 = \sum_{k=1}^{k=\infty} \frac{\beta_k}{k}\, g_1\left({}^a\!\log \frac{k}{b}\right),$$

and by putting ${}^a\!\log b = \alpha$

$$C = \sum_{k=1}^{k=\infty} \frac{\beta_k}{k} \left\{ [{}^a\!\log k - \alpha] + \alpha \right\},$$

where α is an arbitrary quantity. As soon as α is rational, every term in the expansion is rational also. So for instance, for $a = 2$, $\alpha = \frac{1}{2}$ there comes

$$C = -\tfrac{1}{2}\left(-\tfrac{1}{1}\right) + \tfrac{1}{2}\left(\tfrac{1}{2}\right) + \tfrac{3}{2}\left(-\tfrac{1}{3} + \tfrac{1}{4} - \tfrac{1}{5}\right) + \tfrac{5}{2}\left(\tfrac{1}{6} - \tfrac{1}{7} + \ldots - \tfrac{1}{11}\right) +$$
$$+ \tfrac{7}{2}\left(\tfrac{1}{12} - \tfrac{1}{13} + \ldots + \tfrac{1}{22}\right) + \tfrac{9}{2}\left(-\tfrac{1}{23} + \tfrac{1}{24} - \ldots \tfrac{1}{45}\right) + \ldots.$$

To apply another transformation we take in the preceding formula $b = 2$. Then we have

$$C = {}^a\!\log 2 \sum_{k=1}^{k=\infty} \frac{\beta_k}{k} + \sum_{k=1}^{k=\infty} \frac{\beta_k}{k} \left[{}^a\!\log \frac{k}{2}\right],$$

and also

$$C + \log 2 = \sum_{m=0}^{m=\infty} \frac{\beta_{2m+1}}{2m+1} \left[{}^a\!\log \frac{2m+1}{2}\right] + \sum_{k=1}^{k=\infty} \frac{\beta_{2k}}{2k}\, [{}^a\!\log k].$$

Now, if a is any *odd* integer $\beta_{2k} = \beta_k$, therefore we may write

$$\tfrac{1}{2}\, C + \log 2 = \sum_{m=0}^{m=\infty} \frac{\beta_{2m+1}}{2m+1} \left[{}^{a}\log \frac{2m+1}{2} \right],$$

and taking for instance $a = 3$ we will find after some reductions

$$\tfrac{1}{2}\, C + \log 2 = 1 - 8 \sum_{m=1}^{m=\infty} \frac{[{}^{3}\log 3m]}{(6m+1)\,(6m+3)\,(6m+5)}, \quad ([n] = n).$$

5. Returning to the general case $h > 1$, we may try to eliminate as far as possible the irrational terms from the expansions of $C_1, C_2, \ldots$ Then we must apply the formula

$$g_h(x) = \overline{g}_h(x - [x]),$$

and that infers, that we must take either $[n] = n$, $g_1(0) = -\tfrac{1}{2}$, or $[n] = n - 1$, $g_1(0) = +\tfrac{1}{2}$.

Expanding the polynomial $\overline{g}_h(x - [x])$ we have at once

$$g_h(x) = \overline{g}_h(x) + \sum_{m=1}^{m=h} \frac{(-1)^m\, [x]^m}{m!}\, \overline{g}_{h-m}(x),$$

and consequently

$$\overline{W}_h = \frac{C_{h-1}}{\left(\log \frac{1}{a}\right)^{h-1}} = \sum_{m=1}^{m=h} \frac{(-1)^{m-1}}{m!} \sum_{k=1}^{k=\infty} \frac{\beta_k}{k}\, [a\log k]^m\, \overline{g}_{h-m}({}^{a}\log k).$$

In this way the constant C_{h-1} is expressed as an aggregate of the series

$$\sum_{k=1}^{k=\infty} \frac{\beta_k}{k}\, [{}^{a}\log k]^m\, \overline{g}_{h-m}({}^{a}\log k), \quad (m = 1, 2, \ldots, h-1)$$

but as the expansion involves the powers of ${}^{a}\log k$, its terms remain irrational.

Taking for instance $h = 2$, we have

$$\frac{C_1}{\log \frac{1}{a}} = \sum_{k=1}^{k=\infty} \frac{\beta_k}{k}\, [{}^{a}\log k]\, \overline{g}_1({}^{a}\log k) - \tfrac{1}{2} \sum_{k=1}^{k=\infty} \frac{\beta_k}{k}\, [a\log k]^2$$

and as

$$\overline{g}_1({}^{a}\log k) = {}^{a}\log k - \tfrac{1}{2},$$

we may write

$$\frac{C_1}{log\,\frac{1}{a}} = \sum_{k=1}^{k=\infty} \frac{\beta_k}{k}\,[{}^a log\; k]^a\, log\; k - \sum_{k=1}^{k=\infty} \frac{\beta_k}{k} \cdot \frac{[{}^a log\; k]\,[{}^a log\; ka]}{2} \,.\, ([n] = n \text{ or } n-1).$$

Putting $a = 2$, and writing as Mr. HARDY did

$$\left.\begin{aligned} \tau_1 &= \sum_{k=1}^{k=\infty} \frac{(-1)^k}{k}\,[{}^2 log\; k]\, log\; k, \\ \sigma_2 &= \sum_{k=1}^{k=\infty} \frac{(-1)^k}{k}\,\frac{[{}^2 log\; k]\,[{}^2 log\; 2k]}{2}, \end{aligned}\right\} [n] = n,$$

we find

$$-C = \tau_1 - \sigma_2\, log\; 2,$$

according to Mr. HARDY's result [1]).

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[1]) In the definition of the coefficient γ_k Mr. HARDY omitted the factor $\frac{(-1)^k}{k!}$.

Histology. — *"On the Development of the Structure of the Eye-lens in Amphibians"*, by M. W. WOERDEMAN. (Communicated by Prof. L. BOLK).

(Communicated at the meeting of March 29, 1924).

The fibres of the crystalline lens in amphibians do not run straight, no more than in other animals but are curved and specially arranged. This has been examined in detail by C. RABL [1]).

Among the data produced by this author we only mention here that every fibre has two free endings; the one lies in the anterior, the other in the posterior part of the lens. In the anterior hemisphere of the crystalline lens the endings of the lens-fibres come into contact into a vertical plane while in the posterior hemisphere a horizontal plane is formed. The planes extend from the surface to the centre of the lens, gradually decreasing in size. The front plane, namely, is a vertical triangle, the hindmost is a horizontal one.

It struck SPEMANN that the anterior lens-ray coincided with the plane of the foetal eye-fissure. Seeing that the close co-operation between the development of the lens and the optic vesicle has been proved in many experiments with larvae, the question might be raised whether also this phenomenon resulted from a similar co-operation. Prof. SPEMANN suggested to me an experimental research on this problem, for which suggestion I wish to express my obligations to him, as well as for the free access he granted me to his laboratory during this research.

Two ways offered for solution. First of all, before a lens-anlage is visible, the epidermic region from which it is to originate may be excised, rotated and brought to coalescence again. When assuming the disposition of the lens-fibres to depend on the position of the optic vesicle, it may reasonably be expected that the normally located optic vesicle will induce from the rotated epidermic region a normally structured lens.

If, however, the structure of the lens grows through "self-differentiation" from the epidermis, a lens will be engendered of which the anterior ray does not coincide with the plane of the foetal eye-fissure, but is inclined to it.

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[1]) C. RABL, Zeitschr. f. wissensch. Zoölogie. Bnd. 63. Heft 3. 1898.

The second method to be followed was the rotation of the anlage of the eye, while the epidermic region, from which the lens is to arise, is left unchanged.

I acted in either way, but the first proved to be preferable by far from a technical point of view.

The rotation of the future lens-anlage, however, is liable to dangers that might tend to annul the validity of the results. The possible errors are the following: In the first place the larvae are operated upon long before the anlage of the lens has appeared. There are no outward signs then by which it is possible to ascertain which cells of the epidermis are destined to become lens cells. It has, therefore, first to be established where the cell-group lies that will furnish the anlage of the lens, lest the wrong area should be rotated.

The determination of the topographical relations in young larvae I performed by means of local vital staining. To this end small pieces of agar, soaked in a solution of Nile-blue sulphate, are laid for the purpose on a special part of the larval body. The ectoderm cells of this part take up the stain and thus a clearly defined region of the larva is stained vitally. As the stained region remains visible for a long time, its further destiny can easily be traced. This method, suggested by Vogt, yielded very good results. Thus it is also possible to define the area, whose cells are afterwards developed into the crystalline lens.

But even if this area is known, its rotation is yet attended with peculiar difficulties, since the extension of the area in a special larva cannot possibly be distinguished before the first stage of development of the lens. When excising a definite ectodermic region care should be taken to keep the lens-anlage in the centre of it, since only in that case the anlage will not be displaced relative to the optic vesicle.

Fig I*a* represents a larva with a diagram of the optic cup and the lens in their latter position. The anlage of the lens is dotted and divided by a line (the anterior lens ray) into an anterior and a posterior half. Fig. I*b* shows a correct rotation. The upper border of the wound *A* becomes the posterior border, the posterior border *B* becomes inferior border. The lens-anlage is also rotated 90°, but is held in its position relative to the eye-anlage. In Fig. II*a*, however, we see an epidermic region cut out, that has not the lens-anlage for its centre. Rotation of this region induces the position shown in II*b*, from which it is evident that now the anlage of the lens does not correspond anymore with the optic vesicle. Now we have to ask whether normal development of a lens is possible here.

We know from SPEMANN's[1]) researches that in Bombinator larvae

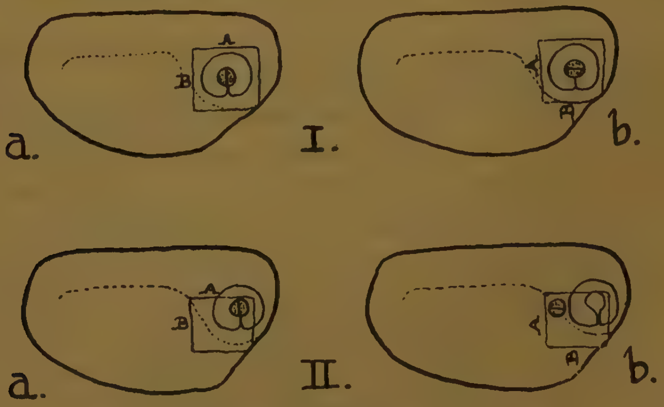

Fig. 1. Schemata (explanation in the text).

the lens-anlage is not competent to differentiate itself, i. e. that the displaced anlage cannot develop into a lens without the presence of the optic vesicle. In case II*b* therefore, it might occur that the displaced anlage of the lens does not develop further after the operation. On the other hand we also know that in Bombinator a lens can be developed from other epidermic regions than are commonly used in normal circumstances, if only they are transplanted over the optic vesicle.

It is conceivable, therefore, that in the case II*b* the ocular cup induces a lens from the epidermic region that has been transplanted over the optic cup. This will give origin to an eye with a lens that has not emanated from the rotated anlage of the lens.

In Rana esculenta the relations are different. In this amphibian the lens-anlage transplanted outside the sphere of influence of the optic cup, can develop into a lens, but it seems that the optic cup is not competent to induce a lens from other epidermic regions.

It follows then that, in Rana esculenta, in the case II*b* an eye without a lens and an ectopic lens may be expected.

I had to bear in mind, therefore, that the result of my experiment would be of value only if it could be established with certainty that after the rotation the lens-anlage would again coincide with the optic cup. Since the anlage of the lens and the optic cup were invisible in the stages operated upon, this certainty could be acquired only through a special technique, by which we were also able to preclude a third error.

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[1]) H. SPEMANN, Zoöl. Jahrb. Bnd. XXXII. Heft 1. 1912.

This third error is the following: After embryonic transplantation it will sometimes occur that the growth of the implantate is slightly slower than that of its environment. This was also possible in the experiment with rotation of the lens, and might lead to a secondary displacement of the implantate, that is pushed away by the more rapidly growing environment.

Considering that the borders of the implantate remain visible only for a short space of time, I had recourse to the following artifice to distinguish the borders for a longer interval. The operations were performed on larvae of Bombinator pachypus and Rana esculenta. Since among these larvae some specimens occur of different pigmentation, two larvae of equal age-period may be taken, the one of which being of a dark, the other of a light pigmentation. The lenticular regions of these larvae are then exchanged and rotated, so that we then have a dark larva with a light implantate and the reverse. Now the borders of the implantate will be visible for some time and we are in a position to ascertain whether the eye-anlage, on becoming visible, really lies beneath the centre of the implantate.

If the larvae, at my disposal, did not differ sufficiently as to pigmentation, I had recourse to vital staining also of some of the larvae with Nile-blue sulphate, while others were not stained. Now, in the operations the lenticular area of a vitally stained larva was substituted by that of an unstained larva of the same age-period, and was also rotated. This again yielded the advantage that the borders of the implantate remained visible for several days running.

For further investigation I discarded all those cases in which the lens-anlage appeared to have shifted from its normal location after the operation, so that I feel assured that the lenses, obtained without exception by the other experimental animals, have indeed originated from rotated lens-anlages.

The stage used for the operations was very young. The neural tube was not yet closed, so that optic vesicles had not taken origin yet. A thickening of the epidermis on the spot of the future lens-anlage is also still totally lacking. Square patches of ectoderm were excised, and subsequently rotated 90°. After the larvae had acquired distinctly visible eye-lenses, they were killed and fixed in MICHAELIS's mixture, stained in toto with borax-carmin and embedded in celloidin-paraffin. The eyes had been cut in such a way that the plane of section runs parallel with the plane of the iris. Hereby the foetal eye-fissure and the direction of the lens-rays become visible in the microscopical section.

Unfortunately the lens is a very difficult object for microscopical examination and I should have killed my experimental animals earlier, for on the whole the lenses were already too hard for suitable sections. In spite of this of the rather large number of experimental animals enough specimens were left to obtain usable preparations. It now appeared that the anterior lens-ray was disposed horizontally in the larvae operated upon, as well in those of Bombinator as of Rana, in other words, that the ray had rotated 90°. This phenomenon is more or less surprising, as it proves that the development of the structure of the lens takes place through "self-differentation", i. e. the environment plays no influence upon it. Now, since in Bombinator the development of the lens depends on the influence of stimuli emanating from the optic cup, one might rather have expected, in the experiments with Bombinator larvae described above, that the optic cup should have induced a normally structured lens from the rotated lens-anlage.

In the Rana-larvae the result comes up to our expectations. When examining the results of the experiments more narrowly it appears now that in the young stages used for operation, the future lenticular structure has already been determined, in other words, that then the direction in which the cells are to grow is established. This is a new fact contributing to our knowledge of the polarity of cells, on which I published a paper some months back [1]).

It also appears that the optic cup has no influence on the direction of the growth of the lens-fibres.

Lastly it follows from the foregoing that, although the optic cup plays a rôle in the development of a lens from a special epidermic region in Bombinator, the structure of that lens is determined by the direction of the growth of its cells, which is independent of the optic cup.

We should, however, emphasize the fact that the relations between lens and optic cup may differ much in two related amphibians. What has been brought forward in this paper holds only for Bombinator pachypus and Rana esculenta.

Amsterdam. *From the Histological Laboratory.*

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[1]) M. W. Woerdeman, These Proceedings, Vol. XXVI, p. 702.

Anatomy. — "*The Chin Problem*". By Prof. L. Bolk.

(Communicated at the meeting of February 23, 1924).

At the meeting of this Academy of February 1921, during which I made a statement concerning the coming into being of specific human characteristics, I supported the hypothesis that all these characteristics bear a common badge, i.e., that they are persistent foetal conditions. As compared with those of the other Primates, man is subject to a checked, partly incomplete course of development. The specifically physical characteristics of man are, therefore, as a rule of a negative nature for, a characteristic that becomes highly developed in the case of other Primates is either entirely absent in man or displays reduced development. I should, however, here point out that this checked development, this foetalisation, — as I have indicated the phenomenon — is not a specifically human characteristic of formation; the influence of this development-governing principle can also be illustrated in other Primates. In man, however, it has become predominantly significant.

In the above-mentioned statement I already pointed out that man possesses two specific physical features which can scarcely be regarded as a symptom of foetalisation, i.e., the projecting chin and the raised bridge of the nose. Neither of these two characteristics can be considered as persisting foetal conditions; they are not of a negative but of a positive nature. During the systematic elaboration of the development of the human form, with the principle of foetalisation as basis, I was faced by the difficulty of explaining these two characteristics in their incipience. And, particularly as far as the chin is concerned, none of the theories anent the creation of this characteristic was capable of solving this difficulty.

This circumstance compelled me to endeavour to arrive at my own conception, based upon my own investigations, in respect of the cause or causes of the existence of these two characteristics. While I am still uncertain as to the nose, the research as far as the chin is concerned has been brought to a conclusion and has led to very surprising results. Of these a short review will be made in this paper, while a detailed account of this research will appear in the course of this year in the transactions of the Academy.

A comparison between the distal end of the lower jaws of an adult man with that of any other Primate teaches us to recognise a difference in the course and direction of the profile of the jaw. In man this is concave, whereas the lower or basal part projects in front of the upper or alveolar part. In the case of the other Primates, the profile is convex and the upper or alveolar part of the jaw projects in front of the lower or basal part.

The question as to the cause of a projecting chin in Man has already occupied many research-workers and the fact that they have arrived at very divergent conclusions proves that the solution of the problem is not so very simple. I shall not enter into the various theories and the very extensive literature on this subject; let it suffice to mention the names: TOLDT, WALKHOFF, WEIDENREICH, SCHWALBE, KLAATSCH, E. FISCHER, KEITH, VAN DEN BROEK, H. VIRCHOW, as some of so many authors who each have supported a more or less individual conception of the genesis of the chin. In this short review I can the more justifiably exclude the theories of these workers from consideration as the results of my own research have no immediate connection with any of these theories. The reason for this is to be found in the circumstance that the morphological basis upon which these theories depend is too limited and, therefore, the manner in which the problem was formulated does not, in my opinion, answer to all the anatomical facts.

From now onwards, a jaw with a profile, as generally exists in the lower Primates, will be indicated as "*Ageneiotic*", whereas the adult human jaw will, in contradistinction to this, be termed "*Eugeneiotic*". Now the formulation of the chin problem by all the authors above-mentioned was simply this: "To what cause must be attributed the development of the eugeneiotic jaw from the ageneiotic, more primitive form?" This simple formulation was the result of a methodologic mistake, viz., the choice of full-grown forms as objects of comparison; the jaw of the full-grown lower Primate was opposed to that of the adult human being. In a treatment on the orbital region of the Primate skull [1]), which appeared a few years ago in the works of this Academy, I already pointed out how necessary it is, instead of making a comparison between full-grown forms — that is, final conditions — to compare the courses of development of those final conditions, as these offer us a broader morphological basis and a more documented manual for the solution of the question as to what causes lie at the bottom of the genesis of the one form from the other. For surely only the growing form is plastic and susceptible to formative causes. Through following this principle during my research into the formation of the chin, I arrived at my conclusion, which differs from that of the abovementioned writers.

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[1]) Verh. Kon. Akad. v. Wet. Deel XX. Amsterdam. 1919.

The individual development of the lower jaw of the lower Primates is very simple. The anlage of the jaw is an ageneiotical one, and this form is maintained during further growth. This may appear from Figure 1, in which the reconstruction of an early stage of development of the jaw of a Mycetes is presented. I shall silently pass over the fact that the profile of the jaw may be unequally inclined backwards in the case of various genera of Primates; the alveolar part projects more in one species than in the other.

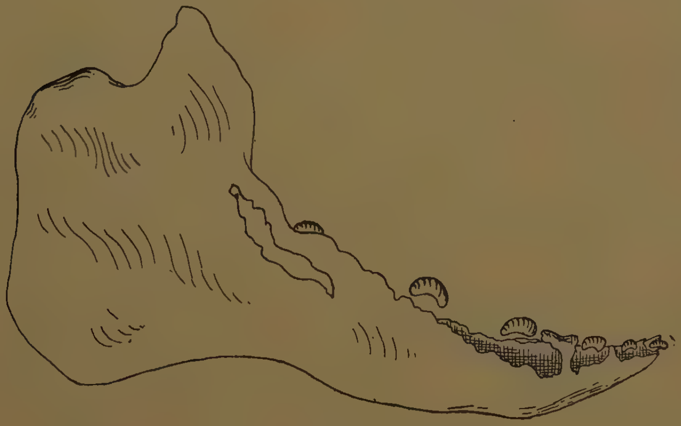

Fig. 1.

With Anthropomorphi a different course of development occurs. As appears from Fig. 2, in which five post-natal stages of development of the lower jaw of Chimpanzee are sketched, the type of the jaw changes during growth. Foetal material for investigation of this group was not at my disposal. The very young Anthropoid possesses a lower jaw whose profile runs straight downwards, a circumstance which may have induced SELENKA's remark that a chin sometimes occurs in the young Gorilla. A projecting chin, as in the case of the adult human being, does not, however, occur. The lower jaw of the young Anthropoid shows more resemblance to that of diluvian man (Spy, la Naulette). This jaw-form cannot be called ageneiotic; it is a type I shall refer to as "*Mesogeneiotic*". Now, it is a very remarkable fact that, in the Anthropoids, this mesogeneiotic type is gradually transformed into the ageneiotic, which is proper to the lower Primates from the beginning. As I am restricting myself to the main lines in this paper, I shall not expatiate upon the details of this transformation.

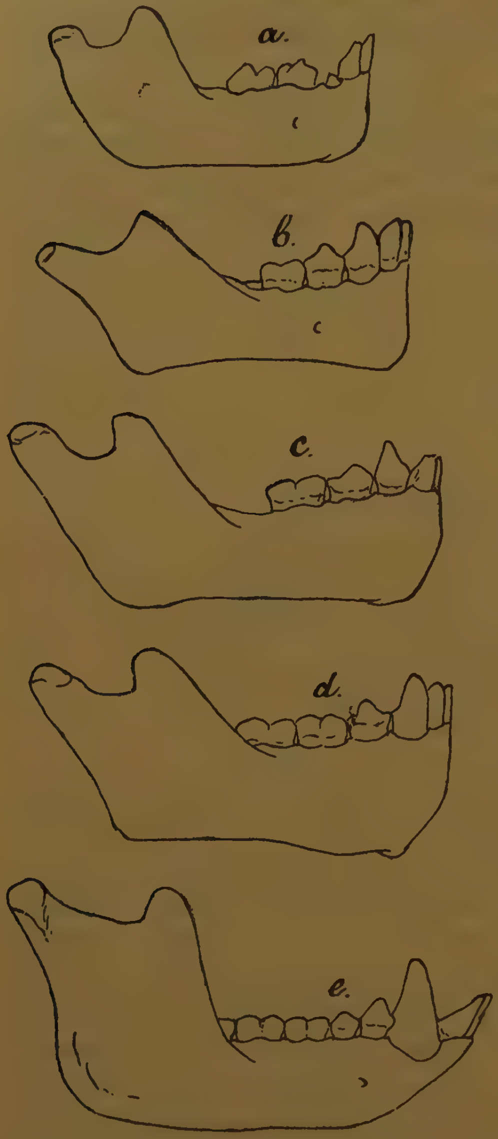

Fig. 2.

The mesogeneiotic type is not always, however, lost. In the Siamanga, for instance, one finds adult individuals in whom this type persists, as can be seen from Fig. 3. The appearance of this form of jaw in the full-grown Siamang is in itself already sufficient to prove that the causes so far suggested of the development of the human form of jaw from the primitive form cannot be correct. For all these causes (reduction of teeth, speech, ossicula mentalia, action of muscles) have this in common that, their formative influence has been at work in the primitive man. And among the Siamangs there are individuals with a lower jaw which, in its profile is even more human than, for instance, the jaw of Heidelberg.

Certainly, the mesogeneiotic jaw with its vertical profile is, as far as its outward appearance is concerned, an intermediate stage between the primitive form of jaw of the lower Primates and the jaw, with its projecting chin, of adult, recent man; aethiologically, however, it is not an intermediate type, for this form came into existence owing to a special cause, entirely different from those which originated the prominent chin.

I established the process of development of the human chin on the basis of reconstructed foetal jaws during the first phases of development and from the numerous jaws of children which are to be found in the anatomical Museum of Amsterdam. The conclusions to which I came differ, not inconsiderably, from the theses so far generally entertained in this regard. Both Toldt and Weidenreich maintain that, in the human embryo the human jaw originally has the shape of that of the lower Primates, the profile tends backwards and downwards and the alveolar part projects, therefore, forwards. In the terminology adopted by me, therefore, the human jaw would have had, in its earliest stage an ageneiotic form. Now, gradually the basal portion would seem to protrude more prominently and finally begins to project in front of the alveolar so that, as Toldt asserts, for instance, the jaw has acquired its typically human aspect in about the second year.

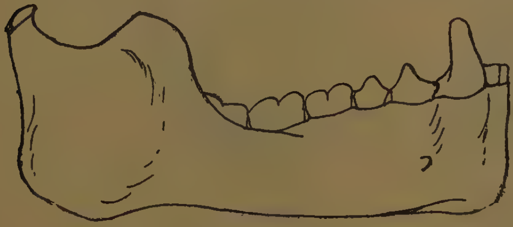

Fig. 3.

Now this conception is not correct. In the process of development and transformation of the mental region of the human jaw, one must distinguish between the origin of two relief-phenomena, viz., of what Klaatsch has called the "Lateral chin" and of the definite chin. In my opinion Klaatsch very justifiably made this distinction. I cannot, however, concur in the explanation given by this writer of the origin of the lateral chin. The lateral chin is formed by both the Tubercula mentalia and the Fossae mentales found above these. These tubercula and fossae come into existence in the earliest stage of the development of the bony jaw and the cause of this will be mentioned briefly later on. In Fig. 4 the lateral surface of the model of a lower jaw of a human foetus of 30 mM. length (vertex to coccyx) is depicted. The Fossa mentalis stretches from the Foramen mentale over the whole distal part of the jaw and the Tuberculum mentale protrudes distinctly from under the Fossa.

Furthermore, this figure proves that Toldt's description of the foetal human jaw is not correct for, if one compares fig. 4 with fig. 1, the difference between the young foetal human jaw and that of the Mycetes' embryo strikes one immediately. The latter is typically ageneiotic but that of the human embryo is mesogeneiotic; its profile slopes, with a slight convex undulation, straight downwards. On comparing the two jaws an impression is obtained as if, in the human being, the entire distal part, from the Foramen mentale upwards, is bent, which would also follow from the different position of the germs of incisivi and caninus in man and Mycetes.

The course of the profile in Fig. 4 is strongly reminiscent of that of the Heidelberg jaw.

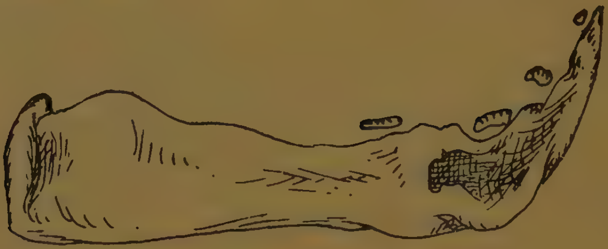

Fig. 4.

So the human jaw is, in its origin, mesogeneiotic. The transition of the frontal border into the lower border of the Dentale is, as appears from Fig. 4, originally rounded off but, through the subsequent development of the Tuberculum mentale, the bend becomes more rectangular. In the human jaw's development, therefore, the primitive form-phase, the ageneiotic jaw of the lower Primates, is missing.

The jaw of the new-born human being is mesogeneiotic but at the same time characterised by the presence of more or less developed Tubercula mentalia, which form the lateral chin. Now, it is certainly a remarkable fact that, in recent man, the mesogeneiotic form of jaw persists until the commencement of tooth-changing, whereas, in diluvian man, this type continued during the entire length of his life.

In Fig. 5 the outlines of the foremost part of some jaws of children of five and six years are sketched. They all have the simple, stretched slope of the mental profile of the jaw. At the ages mentioned a really prominent chin, making the profile of the jaw concave, does not yet exist.

In man, now, the prominent chin develops fairly rapidly and the eugeneiotic type substitutes the mesogeneiotic. This process takes place between the sixth to thirteenth year, as appears from fig. 6 in which the chin is sketched of the jaws of children of eleven and twelve years old. The human jaw, therefore, certainly undergoes a process of transformation during development but not to the extent TOLDT and other writers assert. Moreover, the transformation comes to pass at a later age than that indicated by the writers mentioned. The primitive form is not ageneiotic, to assume the definite eugeneiotic form already in the second year but an mesogeneiotic one, and the transformation into the eugeneiotic type takes place only after the sixth year, that is, therefore, during the change of teeth. Generally both the Tubercula mentalia and the Fossae mentales show a reduction between the second and sixth year — the lateral chin becomes flatter.

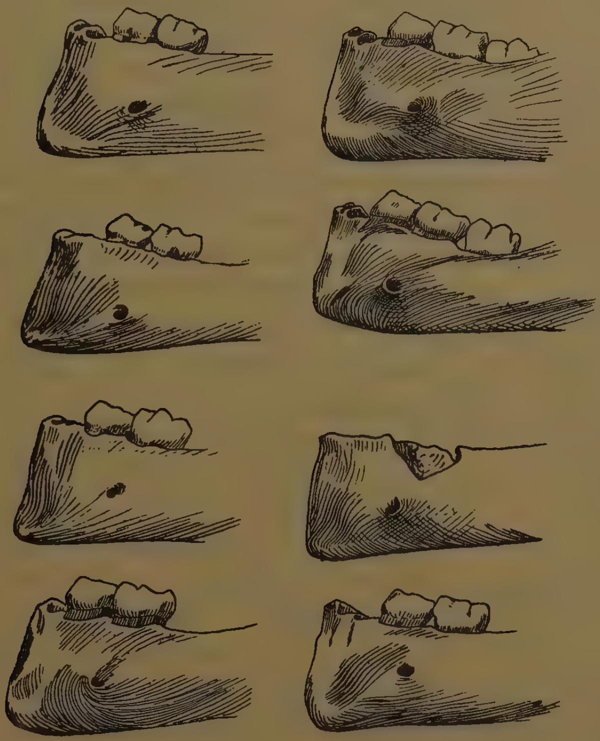

Fig. 5.

From this very succinct summary of the processes of development of the lower jaw in the lower Primates, Anthropoids and man, it will appear that the chin problem is far more complicated than

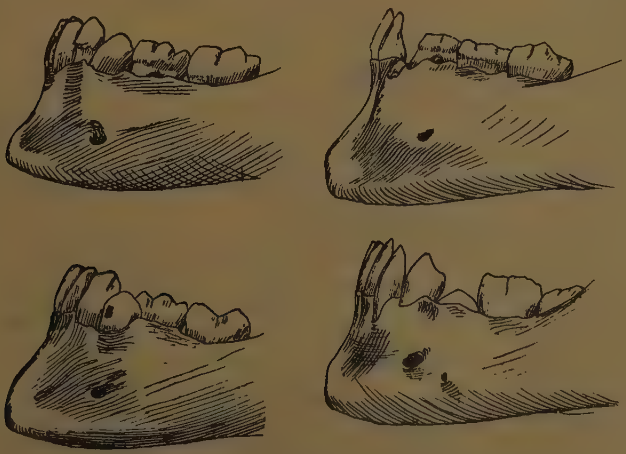

Fig. 6.

has so far been supposed and cannot be epitomised in one single question and answer.

By comparing two final forms — as has been the practice so far — and by the alluding only to the projecting and receding chin, only the following question arose: To what cause should be attributed the development of the former from the latter form? But if, instead of taking final conditions with the greatest differences, one compares processes of development, the number of questions increases owing to the greater quantity of variations of forms and manners of transformation in the different Primates. The nature of these can best be indicated by means of the short summary of the facts in the table given below.

	Embryological form	*Adult form*
Lower Primates	Ageneiotic	Ageneiotic
Anthropoids	Mesogeneiotic	Ageneiotic
Diluvian man	Mesogeneiotic	Mesogeneiotic
Recent man	Mesogeneiotic	Eugeneiotic.

In this table two groups of transformations are expressed, viz., the phylogenetic and the ontogenetic. The former is recognised in the series of adult forms, the latter by comparison of the embryological form with the adult form. The ageneiotic jaw of the adult Anthropomorph is, therefore, a secondary form, the result of operative causes during growth. The nature of these causes will be indicated later on. So there is a remarkable divergence between the processes of development of the lower jaw in man and anthropoids. The infantile type is the same but, whereas the jaw of anthropoids reassumes the more primitive type — so, to put it thus: becomes "more ape-like" —, in man the jaw develops to the highest type and becomes, therefore, "more human".

In the young Anthropoid child, the "lateral" chin is but slightly developed; I suspect that this becomes lost for the most part already during foetal development. This surmise is based upon the manner in which this projection originates, which will be described briefly further on.

The plan set out above represents the starting point of a number of questions; in this short summary I shall restrict myself to answering two of the most important. In the first place, owing to what cause did the mesogeneiotic jaw develop from the ageneiotic and, secondly, what, in man, caused the eugeneiotic form, so typical to adult recent man, to shape itself from the mesogeneiotic? While answering the first question I shall at the same time have an opportunity of throwing some light upon the origin of the lateral chin. All other questions which arise, as well as the many details which came to light during my research, will be discussed at large in my elaborate treatise.

The ageneiotic jaw is characteristic to the lower Primates and must be regarded as the primitive form, for here the embryological form is also of this type. And since, in the human embryo, the jaw is mesogeneiotic, a comparison of the first stages of development must be made between the lower Primates and man to arrive at a conclusion anent the probable reason of the difference in form.

The lower jaw proceeds from both the Dentalia, which develop in the tissue overlying the Meckelian cartilages. Now, if one compares the cartilaginous mandibular arch of the embryo of a monkey — say Mycetes — with that of a human embryo, a considerable difference is immediately noticeable in the relation of the ventral parts of the Meckelian cartilages with respect to each other. In Mycetes the two halves of the arch approach each other under a very sharp hook (Compare Fig. 7), merge into each other and form a fairly long, joint, beak-shaped foremost projection. In the space between the germs of the Caninus and first molar, the arch bends slightly upwards (see Fig. 8).

In man one is faced by a totally different state of affairs. In the first place, the two Meckelian cartilages do not grow together; both halves of the arch remain independent. (Fig. 9). But not altogether.

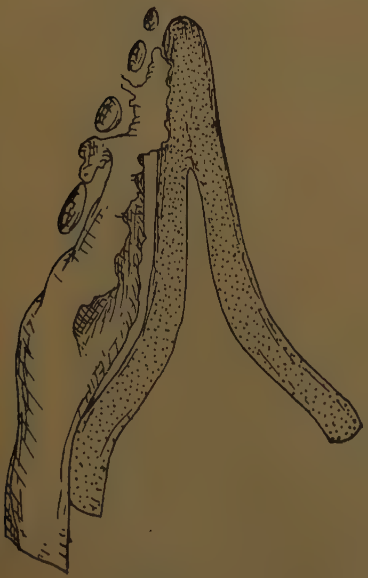

Fig. 7.

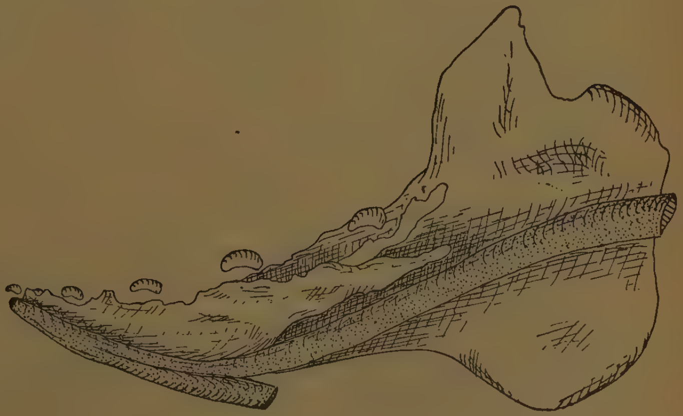

Fig. 8.

For, for a short distance, the medial edges of the distal extremities rest against each other and here a connection does exist between both cartilages.

A second difference is not less obvious. In the space between the germ of the Caninus and the first milk-molar, the Meckelian cartilages bend upwards with a sharp bend, forming a hookshaped curve (see Fig. 10). This peculiarity, which is well known in literature, has never, strangely enough, been connected with the chin problem and yet, as we shall see, it is of the greatest significance in the solution of this problem.

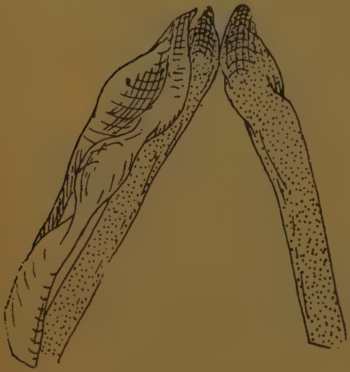

Fig. 9.

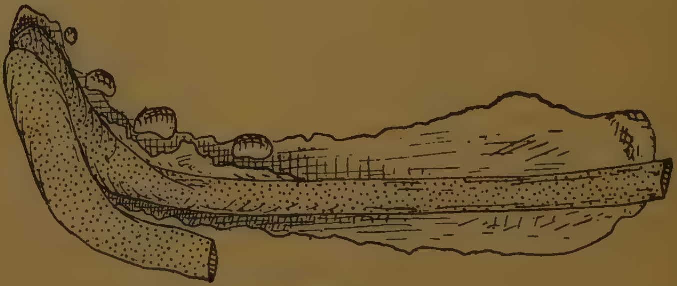

Fig. 10.

In the different shape of the mandibular arch in Mycetes and man, the type of the foetal jaw is, in both forms, as it were preformed. The forward-tapering extremity, with its beak-shaped projection of the but slightly bent arch in Mycetes, answers to the ageneiotic jaw of this monkey, whereas in the hookshaped, upward tending extremities of the cartilages in man, the vertical course of the profile of his foetal and infantile chin is already expressed.

Now does this different structure really exercise any influence upon the form of the osseous jaw? A comparison of fig. 1 with fig. 4 will show that this question must be answered in the affirmative. In complete accordance with the structure of the arch, the osseous jaw of Mycetes is already typically ageneiotic in this early stage of development. In man, it is as if the whole distal end of the Dentale is bent upwards. Not only the Meckelian cartilages but also the Dentalia form a hook-shaped bend. This becomes very apparent if one compares the situation of the dental germs in figures 8 and 10; with Mycetes they are laid in a horizontal plane while in man, on the other hand, they lie more underneath each other. The twisted end of Meckel's cartilages, therefore, causes an analogous curve of the distal extremity of the Dentalia. And this bend of the osseous halves of the jaw in a very early stage of development is the cause of the mesogeneiotic type of the foetal human jaw.

Through this upward bend of the distal ends of the Dentalia, the distal part of the basal border of these pieces of bone in the ageneiotic type becomes the frontal border of the jaw in the mesogeneiotic type. And this can be easily ascertained in the infantile human jaw: the border-edges of the so-called chin-triangle in the infantile human jaw are the upward-tending distal ends of the original basal borders of the Dentalia.

Now the question arises as to why the bend of Meckel's cartilages also forces the Dentalia into such a hook-shaped bend? The cause becomes clear through the topographic and histogenetic relation which in man exist between the cartilaginous mandibular arch and the Dentalia. These conditions are, however, in man quite different from those in Mycetes, as will appear from a comparison between figures 11 and 12. The first-named figure shows the relation of the foremost part of the Dentalia in respect to the beak-shaped projection of the cartilaginous mandibular arch in Mycetes in the level of the second Incisivus. There is no connection between the Dentalia and the cartilaginous arch; the beak-shaped projection lies between both Dentalia. In man an entirely different relationship exists between Dentalia and MECKEL's cartilages for, as follows from fig. 12, here the Dentalia are connected with the lateral surface of the cartilages. This firm union between bone and cartilage extends to the point where Meckel's cartilages bend upwards, i.e., between the germs of the canine and first milk molar or the Foramen mentale. Behind the latter the Dentale develops quite independently and laterally from the cartilage. This intimate relationship between Dentale and

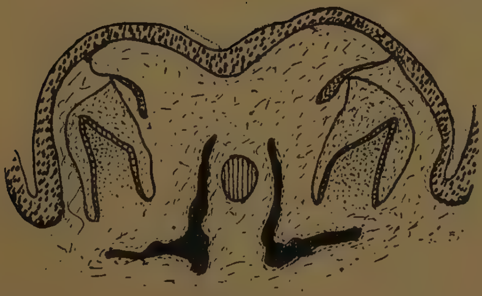

Fig. 11.

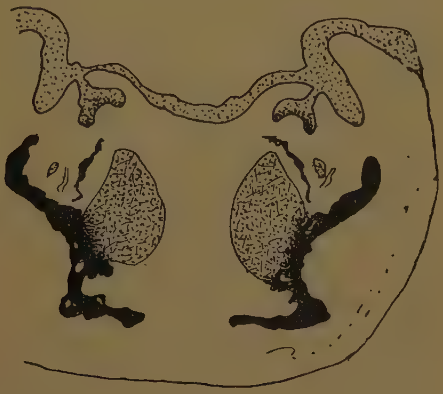

Fig. 12.

cartilage is not secondary; if one examines still earlier stages of development in man, one discovers that the lateral surface of the hook-shaped, twisted part of the cartilage is the starting point of the ossification of the Dentale. Through the upward bend of the distal end of Meckel's cartilages, the same end of the Dentalia is, therefore, also bent in a vertical direction.

Figure 12 may also serve the purpose of demonstrating the origin of the Fossae mentales and of the Tubercula mentalia, which form the lateral chin of man. It appears from this figure that the Fossa mentalis indicates the place where the Dentale is united with Meckel's cartilage, while the Tubercula dentalia form the lowest border of this deeper part, which stretches as far as the Dentale is joined to the cartilage, i.e., to the Foramen mentale.

Thus we have learned the cause of the genesis of the mesogeneitic jaw; this type is already preformed in the cartilaginous mandibular arch. While I shall now silently pass over a number of details, there is one point I should like to mention briefly. The mesogeneiotic jaw of the human child is characterised not only by the straight line of the profile but also by the broad arch of the foremost part. Also this feature of the human mandibel is explained by the behaviour of Meckel's cartilage. If one examines somewhat more developed human embryos, one discovers that the bent, hook-shaped ends of Meckel's cartilages flatten and broaden markedly, so that they become broad cartilage plates, laid in a frontal plane. The Dentalia now grow round the cartilage plates so that the latter are taken up into the jaw, as it were, and take an active part in the formation of the mental region of the jaw, in any case determine its form. The part of Meckel's cartilages behind the Foramen mentale remains a round bar laid medially from the Dentale.

We have now learned a few of the principal points in the morphogenesis of the mesogeneiotic jaw. The question naturally arises: What was the cause of the hook-shaped bending of the distal ends of Meckel's cartilages in man? This question will be briefly dealt with at the end of this paper; we shall now answer the question: Owing to what cause has the highest type of development, namely, the eugeneiotic, in recent man, arised from the mesogeneiotic jaw?

A general indication as to the direction in which the solution of this part of the chin problem must be sought is afforded by the fact that the mentum prominens in recent man does not begin to form itself until after the sixth year, while in the thirteenth a well-developed chin is already in evidence. This fact naturally induces the question as to whether perhaps the origin of the projecting chin may have some connection with the tooth-changing process, which takes place for the most part between the ages mentioned. And indeed, it transpires that this is the case. But not only does a causal connection exist between the transformation of the mesogeneiotic into the eugeneiotic jaw and the tooth-changing process in man, but this same process also explains the transformation, in Anthropoids, of the infantile mesogeneiotic jaw into the ageneiotic of the adult.

In the post-natal development of the set of teeth, two phenomena must be carefully distinguished from one another, viz., the tooth-changing, i. e., substitution of the milk teeth, and the lengthening of the row of teeth by the addition of new elements behind those already present. Moreover, the size of the milk teeth as compared with that of the substituting teeth of the permanent set is of great influence.

Now if, with respect to these points, one compares man with the Gorilla, for instance, the following very important differences are observed: With Gorilla the change of teeth and lengthening take place simultaneously; in man no lengthening of the row of teeth takes place during the changing process. And secondly: the total length of the teeth substituting the milkset is greater than that of the milk-teeth in Gorilla, whereas in man, the length of the milk-set and that of the substituting teeth is the same; indeed, the row of the milk teeth is on the average even a little longer. These differences are of profound significance for the growth of the jaw during the changing process, as will now be demonstrated, in the first place in man.

In literature, the fact that the length of the set of milk teeth in man is equivalent to that of the substituting teeth, is but very little known, though it has been noticed already by Hunter. The significance of this must in the first place be sought in the fact that, after complete eruption of the milk teeth the alveolar arch of the human lower jaw does not grow any more. I was able to confirm the accuracy of this fact, to which TOMES[1]) had already drawn attention, as will appear from the following table. In this table the average of the three following measurements of 75 jaws of children from two to three years old, and of 50 jaws of adults have been stated.

1. The transversal diameter of the dental arch measured from the middle of the hind surfaces of both the second milk-molars, respectively of the front surfaces of both the first permanent molars.

2. The cord of the aveolar arch, measured from one of the points just mentioned to the Incision.

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[1]) Dental anatomy, p. 232 7th edition.

3. The circumference of the alveolar arch, measured from one of two points mentioned on the one side to that on the other.

The results of these measurements are as follows:

	Transversal.	*Cord.*	*Circumference.*
Infants (75)	38.6 m.M.	29.5 m.M.	69.5 m.M.
Adults (50)	39.5 „	28.6 „	66.4 „

From this table it is apparent that the cord and circumference of the adult jaw are even somewhat smaller than those of the infantile jaw. I shall not here enter into the cause of this.

The alveolar arch of the human jaw, therefore, does not grow after the second year. Between the second and sixth year, however, the upper border lengthens somewhat, as room has to be made for the first, permanent molar then being formed. This signifies, therefore, that, between the second and sixth year, the upper border as well as the inferior border of the Corpus mandibulae takes part in the growth in length. Now, however, the phase of tooth-changing follows, which extends from the sixth to the fourteenth year. During this period the alveolar arch of the jaw need not re-form itself, for the permanent set of teeth that now takes its place does not require more room than the milkset, which is substituted by the former. But, neither does a lengthening of the alveolar border occur, for the lengthening of the set by the addition of the second molar only takes place *after* the change of teeth. So, between the sixth and thirteenth year, the growth of the pars alveolaris of the human jaw comes to a stand-still. But the jaw in its entirely *does* grow in conjunction with the increase in size of the body in general and the skull in particular. Also the pars basilaris of the jaw magnifies and lengthens during the period of life mentioned. And the inevitable result must, therefore, now be that this part starts projecting in front of the pars alveolaris and the eugeniotic jaw is formed.

The protruding chin in man is, therefore, the result of the unequal growth of pars alveolaris and pars basilaris between the sixth and fourteenth year. After this period the pars alveolaris also starts lengthening again, to make room for the second and, later third permanent molar.

In the Anthropomorphs, of whom the child also possesses a mesogeneiotic jaw, the phenomena are in every respect opposite to those in man.

In the first place, the alveolar arch expands both in length and breadth, as will appear from the table given hereunder, in which are given the averages of the measurements of six young Chimpanzee jaws with complete milk-set and of seven adult Chimpanzee jaws:

	Transversal.	*Cord.*	*Circumference.*
Infants (6)	30.1 m.M.	31.3 m.M.	75.2 m.M.
Adults (7)	„ 294 .	40.9 „	98.5 „

This enlargement is necessary because the permanent teeth in the Chimpanzee are so much bigger than the milk teeth. Now, as the process of substitution begins after the first permanent molar is cut, the greater room required for the permanent set must be found by lengthening of the alveolar part in front of this tooth, i.e., the actual alveolar arch. This fact is now supplemented by the second, viz., that the process of tooth-changing and the lengthening of the tooth-row take place simultaneously. This appears from the following formula, in which the permanent teeth of the lower jaw are arranged in the rotatory order of their cutting: M_1 I_1 I_2 M_2 P_2 P_1 C. M_3.

Owen and Broca state that M_3 is cut even earlier than C. Thus, while in man M_2 is cut six or seven years after I_2, after tooth-changing is finished, this tooth appears immediately after I_2 with Gorilla. This type of tooth-changing, in which change and increase take place simultaneously, I was able to observe in Macacus and Semnopithecus.

Now, what is the influence of the development of the set of teeth with the Anthropomorphi upon the processus alveolaris? As has already been said, precisely the contrary from that in man. The pars alveolaris must lengthen here considerable in a comparatively short space of time in order to afford sufficient room to the so much larger substituting teeth and the simultaneous increase of the amount of teeth. In man, therefore, there is a stagnation in growth of the upper border during a number of years, whereas with the Anthropoids, on the other hand, there is a greatly accelerated growth, which is closed by the cutting of M_3 and, not uncommonly, by the often simultaneous change of C, when the individual can be regarded as adult. This is reason why, with the Anthropomorphs, the pars alveolaris must lengthen more considerably during growth than the pars basalis. The upper border of the corpus mandibulae begins to protrude in front of the inferior border and the mesogeneiotic jaw of the child is transformed into the ageneiotic jaw of the adult individual.

With this the chief points in the solution of the chin problem are indicated, a solution which was only possible after drawing a sharp border-line to the morphological foundation of the problem. While I now refer to my full treatise on this subject for the details, I shall add one more remark of a more general nature.

Stress has been laid above upon the differences in the post-natal development of the tooth-set in man and the Anthropomorphs. Now it is of the greatest importance for my conception of the development of the human form, mentioned above that the biological character of these differences consists in a retardation of the process of development in man. This retardation had as a consequence that the actual process of tooth.changing and the lengthening of the tooth-set no longer took place simultaneously but occurred successively. And in the previous statement, to which I referred at the beginning of this paper, I laid stress on the fact that the biological basis of the foetalisation principle consists of a checking of human development in its entirety, while this checking in the case of certain characteristics leads to a general suppression. And, as a result of this, the specific features of man are of a negative character. Now it appears that also the form of the human jaw is a result of this process of retardation and that its mentum prominens is but apparently a characteristic with a positive character.

Now, in connection with the shape of his jaw, this fact naturally opens a perspective upon the rate of development of Diluvian man. For his jaw remains mesogeneiotic, that is to say, therefore, that the post-natal development of his set of teeth must have taken place more quickly than that of recent man. On account of this, the necessary lengthening of the pars alveolaris kept pace with the general growth of the jaw. And in this respect it is certainly a very important fact, to which H. Virchow also drew attention in his monography on the subject, that, with the Ehringdorf child, the change of teeth and the lengthening of the set still take place simultaneously. The same was occurring in the childs of Krapina. The second molar is cut before the change of the milk molars — like in Antropomorphi, therefore. From these facts I conclude that Diluvian man must have developed at a quicker rate, and that he must have been full-grown earlier, than recent man, particularly the white races.

Finally, an answer to the question as to why, in man, the distal ends of Meckel's cartilages bend upwards, in which fact the cause lies of the transformation of the ageneiotic into the mesogeneiotic jaw. This bend is the necessary morphological complement of the subcranial position in which the nose capsule persists in the human embryo. Thus also, the transformation of the ageneiotic into the mesogeneiotic jaw is an expression of the general principle of development which endowed man with his specific characteristics. For further details of this, I refer to my extensive paper on this subject.

Mathématiques. — *„Remarques sur certaines congruences linéaires de cubiques gauches.”* Par L. Godeaux, Professeur à l'Ecole Militaire (Bruxelles).

(Présenté par M. le Prof. Jan de Vries dans la séance du 3 Mai 1924).

M. J. de Vries a montré autrefois que la congruence linéaire formée des cubiques gauches passant par cinq points fixes, peut se ramener à une gerbe de rayons, au moyen de la transformation birationnelle définie en rapportant projectivement les surfaces cubiques passant par les arêtes d'un tétraèdre aux plans de l'espace [1]). Vers la même époque, nous avons considéré les congruences linéaires de cubiques gauches s'appuyant en 8 points sur une sextique gauche de genre trois [2]). Pour obtenir ces diverses congruences, il suffit de rapporter projectivement les surfaces cubiques passant par cette sextique gauche aux plans de l'espace; une congruence linéaire de cubiques gauches du type envisagé est alors transformée birationnellement en une congruence linéaire de droites. En particulier, si cette congruence est une gerbe de rayons et si la sextique dégénère en six droites qui sont les arêtes d'un tétraèdre, on retrouve les considérations de M. J. de Vries.

Récemment, M. J. de Vries s'est occupé d'une congruence de cubiques gauches, linéaire, dont les courbes passent par deux points fixes et ont trois bisécantes singulières [3]). Il a précisément montré que cette congruence, étudiée auparavant par M. Stuyvaert [4]), se ramène, par une transformation birationnelle, à une congruence bilinéaire de droites. Cette transformation birationnelle est également un cas particulier de celle que nous avons considérée; ce cas particulier correspond au cas où la sextique se décompose en trois droites et une cubique gauche.

Dans cette note, nous nous proposons de signaler quelques cas particuliers intéressants des congruences linéaires rencontrées dans notre travail cité. Ces congruences présentent des points principaux (communs à toutes les courbes de la congruence); elles s'obtiennent par un procédé analogue à celui qui a été utilisé par M. J. DE VRIES dans sa dernière note citée.

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[1]) *Congruences of twisted curves in connection with a cubic transformation*, Kon. Akademie v. Wetensch. Amsterdam, Proced., 1908, t. XI, p. 84.

[2]) *Nouveaux types de congruences linéaires de cubiques gauches.* Nouvelles Annales de Mathématiques, 1909.

[3]) *Représentation of a bilinear congruence of twisted cubics* ... Kon. Akad. v. Wetensch., Amsterdam, Proced. 1922, t. XXV, p. 22.

[4]) *Etude de quelques surfaces* ... Gand, Hoste, 1902.

1. Soit C_6 une sextique gauche de genre trois[1]). Elle appartient, comme on sait, à ∞^3 surfaces cubiques F_3 formant un système linéaire. Deux surfaces cubiques F_3 ont encore en commun une cubique gauche γ_3 s'appuyant en 8 points sur C_6. Trois surfaces cubiques F_3, n'appartenant pas à un même faisceau, ont en commun un et un seul point en dehors de la courbe C_6.

Désignons par Σ l'espace ordinaire contenant C_6 et rapportons projectivement les surfaces cubiques F_3 passant par C_6 aux plans d'un second espace ordinaire Σ'. Nous obtenons ainsi une correspondance birationnelle entre les espaces Σ et Σ', bien connue, étudiée par CREMONA, CAYLEY, NOETHER, STURM,... Aux plans de Σ correspondent, dans Σ', des surfaces cubiques F'_3 passant par une courbe C'_6, d'ordre six et de genre trois. Les points fondamentaux de la correspondance sont, dans Σ, les points de C_6, dans Σ', les points de C'_6.

A un point de C'_6 (ou de C_6) correspond, dans Σ (ou Σ'), l'ensemble des points d'une trisécante de C_6 (ou de C'_6). Les trisécantes de C_6 (ou C'_6) forment une surface d'ordre huit, passant trois fois par la courbe C_6 (ou C'_6).

Ces points étant rappelés, soit G' une congruence linéaire de droites dans Σ'. En général, à une droite de G' correspond une cubique gauche γ_3 de Σ. Aux différentes droites de G' correspondent, dans Σ, les cubiques gauches d'une congruence G évidemment linéaire. Nous allons examiner les diverses congruences linéaires G que l'on peut obtenir par ce procédé. Dans ce but, nous utiliserons la propriété suivante: Une courbe d'ordre n de Σ', s'appuyant en a points sur C'_6, a pour transformée dans Σ, une courbe d'ordre $3n-a$, s'appuyant en $8n-3a$ points sur C_6.

2. *Congruences du premier type.* Supposons tout d'abord que G' soit une gerbe de rayons de sommet P'. Observons que P' ne peut se trouver sur C'_6, car aux rayons de G' correspondraient, dans Σ, des cubiques gauches dégénérées en une trisécante fixe de C_6 et en des coniques variables (formant une congruence linéaire).

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[1]) Pour une étude systématique de la courbe gauche d'ordre six et de genre trois, on peut consulter l'ouvrage de M. STUYVAERT, *Cinq Etudes de Géométrie Analytique.* Gand, Van Goethem, 1908.

Nous aurons donc à examiner deux hypothèses :

a. P' ne se trouve pas sur la surface des trisécantes de C'_6;

b. P' se trouve sur une trisécante de C'_6.

Dans la première hypothèse, il correspond à P', dans Σ, un point P bien déterminé, et la congruence G, transformée de G', est formée par les cubiques gauches γ_3 passant par P. Cette congruence G a été étudiée par M.M. Veneroni et Stuyvaert. On en trouvera la bibliographie, ainsi que l'étude de plusieurs cas particuliers, dans la dissertation de M. J. de Vries junior [1]).

Plaçons-nous dans la seconde hypothèse et soit a' la trisécante de C'_6 passant par P', A le point de C_6 qui correspond à cette droite a'. A une droite d' passant par P' correspond dans Σ une cubique gauche γ'_3 passant par A; en général, γ'_3 ne sera pas tangente à la courbe C_6 en A. Soit π le plan tangent commun à C_6 et à γ'_3 en A. Aux ∞^1 surfaces cubiques F_3 passant par γ'_3 et par suite tangentes à π en A, correspondent dans Σ', ∞^1 plans formant un faisceau d'axe d'. Toute droite de la congruence G' est située sur un plan de ce faisceau, par suite, les cubiques gauches de la congruence G correspondante sont tangentes en A au plan π.

Ces cubiques gauches s'appuient en sept points variables sur C_6, donc :

Les cubiques gauches s'appuyant en huit points sur une sextique gauche de genre trois et touchant en un de ces points, supposé fixe, un plan tangent à la sextique, forment une congruence linéaire.

3. *Congruences du second type.* Supposons maintenant que G' soit formée par les cordes d'une cubique gauche Γ'_3 s'appuyant en α points sur la courbe C'_6. La congruence correspondante est formée par les cubiques gauches γ_3 s'appuyant en 8 points sur C_6 et en deux points sur la transformée de Γ'_3. Celle-ci est une courbe $\Gamma_{9-\alpha}$ d'ordre $9-\alpha$ s'appuyant en $24-3\alpha$ points sur C_6. On a donc $\alpha \leqq 8$, et la courbe $\Gamma_{9-\alpha}$ est au moins d'ordre un.

En particulier, pour $\alpha = 8$, on obtient une congruence linéaire de cubiques gauches ayant une bisécante fixe.

4. *Congruences du troisième type.* Supposons enfin que G' soit formée par les droites s'appuyant en un point sur une droite Γ'_1 et en un second point sur une courbe Γ'_n, d'ordre n, s'appuyant elle-même en $n-1$ points sur Γ'_1. Soient de plus α_1 le nombre de points communs à Γ'_1 et C'_6, α_2 le nombre de points communs à

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[1]) *Bilineaire congruenties van kubische ruimtekrommen.* Utrecht, 1917.

Γ'_n et C'_6. Supposons tout d'abord $\alpha_1 < 3$, $\alpha_2 < 3n$. Alors, la congruence G transformée de G' est formée des cubiques gauches γ_3 s'appuyant en huit points variables sur C_6, en un point sur la courbe Γ_1, d'ordre $3-\alpha_1$, transformée de Γ'_1, et en un point sur la courbe Γ_2, d'ordre $3n-\alpha_2$, transformée de Γ'_2. La courbe Γ_1 s'appuie en $8-3\alpha_1$ points sur C_6 et la courbe Γ_2 en $8n-3\alpha_2$ points. Les courbes Γ_1, Γ_2 ont au plus $n-1$ points communs; ce nombre est diminué d'une unité chaque fois qu'un des points communs à Γ'_1, Γ'_2 appartient à C'_6. En supposant que le nombre de ces points, soit α, la configuration des lignes singulières de G peut être résumée par le tableau suivant:

	γ_3	C_6	Γ_1	Γ_2
γ_3		8	1	1
C_6	8		$8-3\alpha_1$	$8n-3\alpha_2$
Γ_1	1	$8-3\alpha_1$		$n-\alpha-1$
Γ_2	1	$8n-3\alpha_2$	$n-\alpha-1$	
Ordres	3	6	$3-\alpha_1$	$3n-\alpha_2$

$\alpha_1 < 3$,

$\alpha \leqq \alpha_2 < 3n$.

Supposons $\alpha_1 = 3$, $\alpha_2 < 3n$. Alors, à la courbe Γ'_1 correspond un seul point A de C_6 et toutes les courbes de la congruence G passent par ce point A; elles rencontrent encore C_6 en sept points variables. Elles s'appuient en un point sur la courbe Γ_2 qui correspond à Γ'_2. Cette courbe Γ_2 est d'ordre $3n-\alpha_2$ et passe en général $n-1$ fois par le point A. Ce nombre est abaissé d'une unité pour chaque point commun à Γ'_1, Γ'_2, appartenant à C'_6.

Les cubiques gauches passant par un point fixe d'une courbe gauche d'ordre six et de genre trois, s'appuyant encore en sept points variables sur cette courbe et s'appuyant en un point sur une courbe rationnelle d'ordre $3n-\alpha_2$ *passant* $n-\alpha-1$ *fois par le point fixe* ($\alpha \leqq \alpha_2$), *forment une congruence linéaire.*

On remarquera que les $n-1$ points d'appui de Γ'_1 sur Γ'_2 peuvent se composer de quelques points multiples pour la courbe Γ'_2. En particulier, si Γ'_2 est une courbe plane, il y aura un point $(n-1)$-uple.

Supposons $n = 1$, $\alpha_1 = \alpha_2 = 3$. La congruence G' est formée des droites s'appuyant sur deux trisécantes de C'_6. Les cubiques de la congruence G correspondante passent par deux points fixes de C_6.

Les cubiques gauches s'appuyant sur une sextique de genre trois en deux points fixes et six points variables, forment une congruence linéaire.

5. On peut envisager le cas où, dans la congruence G' considérée plus haut (n°. 4), la courbe Γ'_2 est formée de n droites infiniment voisines de Γ'_1. On obtient sans difficulté les congruences G correspondantes [1]. Bornons-nous à singuler la congruence suivante: Si la congruence G' est formée des droites s'appuyant sur deux trisécantes infiniment voisines de C'_6, les courbes γ_3 correspondant à ses droites seront tangentes à la courbe C_6 en un point fixe, donc:

Les cubiques gauches tangentes à une sextique gauche de genre trois en un point fixe et s'appuyant en six points variables sur cette courbe, forment une congruence linéaire.

6. La plupart des propriétés des congruences de cubiques gauches rencontrées dans ce travail s'obtiennent aisément au moyen des propriétés de la transformation birationnelle utilisée. Ainsi, toute surface engendrée par ∞^1 cubiques gauches de G sera transformée en une surface appartenant à la congruence de droites G'. Nous ne nous arrêterons pas à ces développements, nous nous limiterons à l'étude du problème suivant: Etant donné une congruence linéaire G formée des cubiques gauches s'appuyant en 8 points sur la courbe C_6, en un point sur une cubique gauche Γ_1 (s'appuyant elle-même en 8 points sur C_6) et en un point sur une courbe Γ_2; d'ordre $3n$ (s'appuyant en $8n$ points sur C_6 et en $n-1$ points sur Γ_1), déterminer l'ordre de la surface lieu des points d'appui sur les cubiques de G des bisécantes passant par un point fixe Q.

Désignons par Φ la surface considérée, par m sur ordre, par μ la multiplicité de C_6 pour Φ, par μ_1, μ_2 les multiplicités respectives de Γ_1, Γ_2 pour Φ, par ν la multiplicité du point Q pour cette surface Φ.

Soient Q', Γ'_1, Γ'_2 le point, la droite et la courbe d'ordre n (s'appuyant en $n-1$ points sur la droite Γ'_1) qui correspondent respectivement dans Σ', au point Q et aux courbes Γ_1, Γ_2. La transformée de la congruence donnée G sera la congruence G' formée par les droites s'appuyant sur Γ'_1 et Γ'_2.

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[1] Pour une congruence linéaire de cubiques gauches obtenue par ce procédé, voir notre note *Sur la sixième congruence de cubiques gauches de* M. STUYVAERT, Bull. Acad. R. de Belgique, 1909.

A la surface Φ correspond dans Σ' la surface Φ' lieu des points d'appui des bisécantes des cubiques gauches s'appuyant en 8 points sur C'_6 et passant par Q', faisant partie de G'. L'ordre de la surface Φ' est $3m-8\mu$. Cette surface passe $m-3\mu$ fois par la courbe C'_6, μ_1 fois par Γ'_1, μ_2 fois par Γ'_2, ν fois par le point Q'.

Il y a $1+3n$ droites de G' bisécantes d'une cubique gauche, donc une cubique passant par Q' et s'appuyant en 8 points sur C'_6 rencontre Φ' en $2(1+3n)$ points en dehors de C'_6 et de Q'. On a donc

$$3(3m-8\mu)=8(m-3\mu)+\nu+2(1+3n),$$

c'est-à-dire

$$m=\nu+2(1+3n)\quad.\ .\ .\ .\ .\ .\ .\ .\quad(1)$$

Il y a une seule cubique passant par Q' et s'appuyant 8 fois sur C'_6 qui soit bisécante d'une droite de G', donc on a

$$\mu_1+\mu_2+2=3m-8\mu\quad.\ .\ .\ .\ .\ .\ .\ .\quad(2)$$

Soient maintenant H la gerbe de rayons de sommet Q, K le système des cubiques gauches passant par un point P et s'appuyant en 8 points sur C_6, H' et K' les systèmes de cubiques gauches et de droites correspondant respectivement à H, K dans Σ', P' le point correspondant à P. Désignons par Ψ la surface lieu des points d'appui des cordes des cubiques de K passant par Q, par Ψ' la transformée de Ψ dans Σ'. Ψ est le lieu des points d'appui des cordes des cubiques de H' passant P. Donc Ψ et Ψ' possèdent les mêmes ordre et multiplicités le long de C_6, C_6', en P, Q' et en Q, P'. Si m' est l'ordre de Ψ, μ' la multiplicité de C_6 pour Ψ, ν'_1 la multiplicité de P, ν'_2 celle de Q pour Ψ, on a donc

$$3\,m'-8\,\mu'=m',\quad m'-3\,\mu'=\mu',\quad \nu'_1=\nu'_2,$$

d'où $m'=4\,\mu'$, $\nu'_1=\nu'_2=\nu'$.

La cubique de H passant par Q est une courbe simple de Ψ d'après la définition de cette surface; par suite la droite $P'Q'$ est simple pour Ψ', ce qui exige

$$2(\nu'-1)\leqslant m'-1.$$

Observons de plus qu'une droite passant par Q est la corde d'une seule cubique de K, donc $m'=\nu'+2$. On a donc $\nu'\leqslant 3$, $m'\leqslant 5$. Comme d'autre part on a $m'=4\mu'$, il vient nécessairement $m'=4$, $\mu'=1$, $\nu'=2$. Ainsi la surface Ψ est d'ordre 4, passe simplement par C_6 et P, Q sont des points doubles pour Ψ (ce résultat était d'ailleurs connu). Si en particulier nous considérons la section de Ψ

par un plan passant par Q, nous voyons que le lieu des points d'appui des cubiques gauches s'appuyant en 8 points sur C_6 et passant par un point, sur les cordes qui font partie d'un faisceau-plan, est une courbe d'ordre 4 passant doublement par le sommet du faisceau.

Revenons à la surface Φ'. La section de cette surface par un plan π passant par Γ'_1 sera, d'après le résultat précédent, une courbe du quatrième ordre passant doublement par le point de rencontre du plan π et de Γ'_2. Cette courbe passera simplement par les points communs à π et à C'_6. On en conclut que $m-3\mu=1$, $\mu_2=2$, $\mu_1=3m-8\mu-4$, c'est-à-dire, en utilisant (1) et (2),

$$m=3\mu+1,\ \mu_1=\mu-1,\ \mu_2=2,\ \nu=3\mu-6n-1.$$

Si nous considérons en particulier la section de Φ' par le plan π passant par Γ'_1 et Q', nous obtenons une courbe du quatrième ordre passant deux fois par Q'. Donc $\nu \leq 2$. D'autre part, d'après la dernière égalité écrite ci-dessus, $\nu+1$ doit être multiple de 3, donc $\nu=2$. On en déduit

$$\mu=2n+1,\quad m=6n+4,\quad \mu_1+2n,\quad \mu_2=2.$$

Par suite: *La surface* Φ *est d'ordre* $6n+4$, *passe* $2n+1$ *fois par* C_6, $2n$ *fois par* Γ_1, 2 *fois par* Γ_2 *et* 2 *fois par le point* Q.

7. La transformation birationnelle utilisée dans ce travail peut fournir d'autres congruences linéaires de cubiques gauches. En voici un exemple.

Considérons, dans Σ', une droite Γ'_1 ne rencontrant pas C'_6 et une courbe Γ'_2 d'ordre n, s'appuyant en $n-1$ points sur Γ'_1, en α points sur C'_6.

Soit π' un plan passant Γ'_4. Considérons les quintiques du plan π' ayant un point double ordinaire en chacun des points de rencontre de π' et C'_6 et un point simple au point de rencontre de π et de Γ'_2. Ces courbes ne sont pas dégénérées, car il n'existe aucune conique s'appuyant en six points sur C'_6. Ces courbes forment un faisceau et lorsque π' tourne autour de Γ'_1, on obtient une congruence linéaire de quintiques planes.

A une de ces quintiques correspond, dans Σ, une courbe d'ordre $3\times 5-2\times 6=3$, qui est rationnelle. Cette courbe est une cubique gauche, car la quintique envisagée ne se trouve pas sur une surface cubique circonscrite à C'_6. Si l'on désigne par Γ_1 la cubique gauche correspondent à Γ'_1, par Γ_2 la courbe qui correspond à Γ'_2, on obtient le résultat suivant:

Les cubiques gauches s'appuyant en quatre points sur une sextique gauche de genre trois, en cinq points sur une cubique gauche *s'appuyant elle-même huit fois sur la sextique, et en un point sur une courbe d'ordre* $3n-\alpha$, *s'appuyant en* $8n-3\alpha$ *points sur la sextique et en* $n-1$ *points sur la cubique, forment un congruence linéaire.*

C'est une congruence que nous avions obtenue antérieurement par deux autres voies [1]).

Si l'on considère les quintiques ayant six points doubles sur C'_6, situées dans les plans passant par Γ'_1 et passant de plus par un point fixe de cette droite, on obtient le résultat suivant:

Les cubiques gauches s'appuyant en quatre points sur une sextique gauche de genre trois et en quatre points variables et un point fixe sur une cubique s'appuyant elle-même en 8 points sur la sextique, forment une congruence linéaire.

Bruxelles, 23 avril 1924.

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[1]) *Sur la quatrième congruence de cubiques gauches de* M. Stuyvaert (Nouvelles Annales de Math., 1911). *Détermination des congruences linéaires de cubiques gauches s'appuyant en cinq points sur une cubique gauche fixe* (Rend. Circ. Matem. di Palermo, 1911, XXXII).

Chemistry. — *"On Disaccharins."* (Preliminary communication). By Prof A. F. HOLLEMAN and H. J. CHOUFOER.

(Communicated at the meeting of March 29, 1924).

Though the investigation of saccharin derivatives is already pretty extensive, there are still three directions in which further researches might be made, in order to obtain a better insight in the relation between structure and taste of these interesting compounds. [1])

Researches in the third direction indicated loc. cit., i.e. into the taste of compounds containing the group CO—NH—SO_2 more than once, substituted at orthoplaces in the benzene nucleus, have been taken in hand by me (Ch.). As also POLLAK and LUSTIG [2]) have undertaken researches in this region, I feel obliged to make a provisional communication on the results already found by me.

My purpose was to prepare disaccharins which can be derived from the three xylenes; this rendered a renewed study of their di-sulphonic acids necessary.

Disaccharins from m-xylene.

Through WISCHIN'S [3]) investigations a disulpho-chloride, melting-point 129°, has become known. The corresponding di-sulphonic acid is formed by heating m-xylene with crystallized fuming sulphuric acid.

When I applied chloro-sulphonic acid for the sulphonation, I found that on treatment of m-xylene with a great excess, the disulpho chloride is immediately formed. On pouring out on ice, a pretty pure product arises, which is only contaminated by small quantities of an oily substance. The oil can be easily removed with ether, after which through recrystallisation from ligroin, ether, and chloroform, the substance can be obtained pure, melting-point 129°.

From the disulpho-chloride the diamide can be obtained with alcoholic ammonia in a fair yield. It is advisable to dissolve the disulpho chloride first in benzene, and to add the calculated quantity of alcoholic ammonia in drops amidst stirring. There is formed a white crystalline mass. By washing with a little water the NH_4Cl can be easily removed. Then the amide is recrystallized from much water. It appears in long, fine needles, melting-point 249°.

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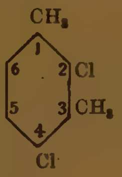
[1]) Rec. **42**, 839 (1923).

[2]) A. **433**, 191 (1923).

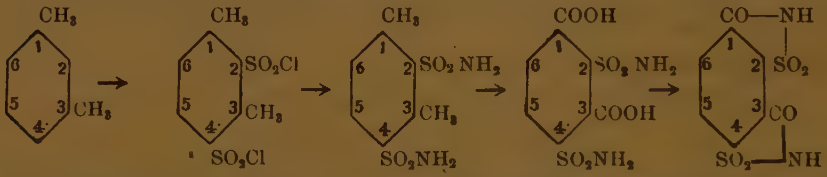
[3]) Ber. **23**, 3113 (1890).

Wischin [1]) has already oxidized this diamide with $KMnO_4$ in aqueous solution. He states that he has obtained a diimide of the m.p. 225°. On repetition of this oxidation I observed that when the diamide is oxidized with $KMnO_4$, and the solution is acidified, first unchanged diamide crystallizes out, and then a finely crystallized product m.p. 268°—270°. According to analysis this agrees with a diimide, called *disaccharin* by us.

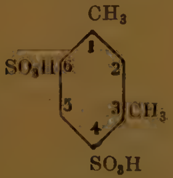
Wischin found that the disulpho chloride m.p. 129° on strong heating with PCl_5 passes into a dichloro-xylene of the structure:



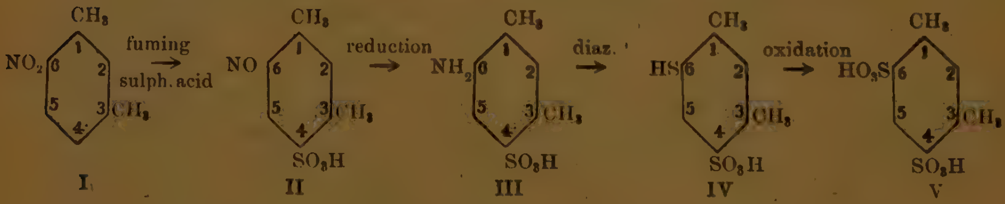
This induces us to represent the conversions described as follows:



It is easy to see that it must be possible to obtain from the sulphonic acid a second disaccharin.



To prepare this substance the following plan was devised:



The nitroxylene was obtained from Poulenc frères. With fuming sulphuric acid it is converted into the sulphonic acid (II) at the ordinary temperature. With $BaCO_3$ it is converted into the Ba-salt. This being sparingly soluble, the precipitation of $BaSO_4$ must be boiled out a many times. The Ba-salt is converted into the Na-salt with soda. This Na-salt crystallizes in fine needles. It is reduced by boiling with $(NH_4)_2S$. The free amido sulphonic acid (III) is sparingly soluble.

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[1]) Ber. 23, 3113 (1890).

The most convenient way of bringing about the diazotation is by mixing the Na-salt of the sulphonic acid (III) with the equivalent quantity of Na-nitrite, and slowly adding drops of hydrochloric acid at 0°.

The diazonium solution obtained is added to a solution of sodium disulphide. The reaction can be tempered by the addition of some ice. Already at 5° there takes place a violent generation of nitrogen.

The solution now contains the Na-salt of IV, which by oxidation at low temperature with potassium permanganate passes into the Na-salt of the disulphonic acid V. This is purified by evaporation and recrystallisation, but is not yet free then from inorganic salts. Now the mass is treated with PCl_5. On pouring out on ice the inorganic substances goes into solution and the sulpho-chloride is obtained pretty pure. It crystallizes very beautifully from chloroform, and melts at 128°. The yield is satisfactory.

These experiments were nearly completed when I became acquainted with the thesis for the doctorate of Pfannenstill [1]). It appeared from this that he had prepared the disulpho-chloride in a similar way. According to him the melting-point is 131°. He has sulphonised the xylidine. His method of diazotation [2]) does not proceed so smoothly as mine (temp. ± 70°).

By recrystallization from chloroform I could bring the melting-point of my compound at 129°. Pfannenstill prepared the diamide by treatment with ammonia. It crystallized in small globules, m.p. 243°. With alcoholic ammonia I could convert the dichloride into the diamide, which crystallized from water in shiny needles, m.p. 248°. On oxidation with permanganate I obtained the unchanged product back, after having acidified the liquid. From the mother liquid I could isolate a hygroscopic nitrogen-containing product, of which the percentage of N is 7.8 (N-percentage of a disaccharin is 9.7).

Then appeared the before-mentioned paper by Pollak and Lustig. By synthesis they too succeeded in preparing the 1,3-xylene-4,6-disulpho-chloride. They also showed that this disulpho chloride is identical with the disulpho-chloride, obtained by direct sulphonation from m-scylene.

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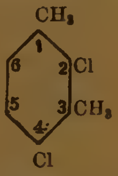
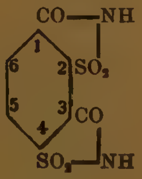
[1]) Diss. Lund. 1894.

[2]) P. Klason, Ofvers. af K. Vet. Akad. förh. 1887.

On comparison also my product appeared to be identical with it. On repetition of the oxidation of the diamide with larger quantities, I now could isolate the disaccharin, m.p. 268° by the side of the unchanged product. It is now evident that on direct sulphonation of m-xylene, 1,3-xylene-4,6-disulphonic acid is formed, and that the saccharin prepared from it has the corresponding structure. The melting-point of this substance is 268°—270°; its taste is acid and bitter.

Wischin has shown that by heating 1,3-xylene-4,6-disulpho-chloride with PCl_5, the compound is  formed. We must therefore, assume a shifting of groups to have taken place. Such an assumption is always more or less arbitrary.

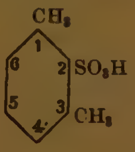
Now the saccharin



is still to be prepared from m-xylene.

This comes chiefly to the same thing as the preparation of 1,3-xylene-2,4-disulphonic-acid.

The two mono-sulphonic acids



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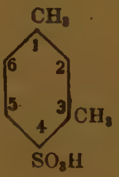


has long been known.

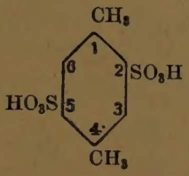
Pfannenstill has shown that on further sulphonation 4-sulphonic acid produces 4,6-disulphonic acid.

He calls this 2,4-disulphonic acid, in connection with Wischin's experiments. The 2-sulphonic acid yields chiefly 4,6-disulphonic acid on continued sulphonation, and as bye-product a disulphonic acid, the chloride of which is oily. In this sulphonation there has, therefore, again taken place a shifting of groups.

To all probability the oily disulpho chloride is the 2,4-derivative so that it must be possible to prepare the second disaccharin from it. The amide has already been prepared by me; it melts at ± 220°.

Disaccharins from p-xylene.

For the preparation of these saccharins are required the sulphonic acids:



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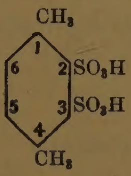
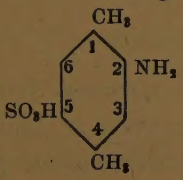


It has been found by Jesse Holmes[1]) that on direct sulphonation of *p*-xylene 1,4-xylene-2,6-disulphonic acid is formed. The di-amide prepared from this cannot be oxidized to a di-imide with $KMnO_4$.

Pfannenstill has found that on sulphonation with SO_3HCl there is formed besides 1,4-xylene-2,6-disulphochloride, m.p. 75°, a small quantity of another disulpho-chloride, m.p. 160°. These experiments have also been made by Pollack and Lustig, who were evidently not acquainted with Pfannestill's work.

For the preparation of 1,4-xylene-2,5-disulphonic acid I have started from *p*-xylene. This was nitrated. The nitro-compound was reduced with iron-filings, and the xylidine was converted into the sulphonic acid with fuming sulphuric acid. This sulphonic acid has the structure  as *p*-xyloquinone is formed on oxidation with chromic acid.

The Na-salt of the amido sulphonic acid is mixed with the quantity of Na-nitrite calculated and while it is being cooled, drops of hydrochloric acid are added. The sparingly soluble diazonium hydroxide precipitates for the greater part.

In the meantime a solution of sodium disulphide is prepared, which is cooled by ice being thrown into it. Now the diazonium hydroxide is added. It goes into solution and gives an intensive colour to the liquid. At once a gradual generation of nitrogen sets in. Finally it is heated on the waterbath, till the nitrogen generation stops. Then the solution has lost its intensive colour.

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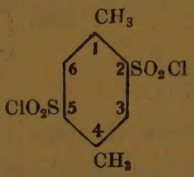
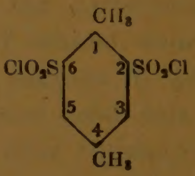
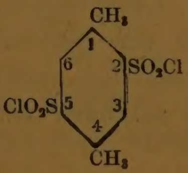
[1]) Am. 13, 371 (1891).

Now permanganate solution is added while the cooling is continued, till it is no longer immediately decolorized. The liquid is entirely evaporated, and the salt mass is dried and powdered. This is shaken with its own weight of PCl_5. At once a violent generation of heat sets in. When the reaction has stopped, the substance is heated to 140°, at which temperature $POCl_3$ goes over. Now the product is placed on ice. A solid mass of a somewhat brownish colour separates, which is filtered off. A little ether almost entirely removes the brown colour The disulpho chloride, of which the following structure was ascertained by this synthesis proved to be identical with the product of the direct sulphonation of p-xylene.



Hence in the sulphonation of p-xylene is formed as main product, m.p. 75°, and as bye-product, m.p. 162°.

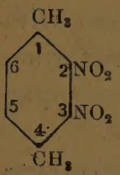




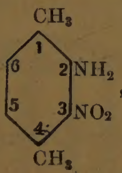
In the usual way the disulphamide is obtained from the disulpho chloride. The disulphamide is insoluble, even in boiling water, but can be purified by solution in lye, and subsequent precipitation by means of hydrochloric acid; m.p. ± 310°.

For the preparation of disaccharin the diamide is stirred up in water, and oxidized with $KMnO_4$ on a waterbath while heated. After the oxidation the manganese dioxide is filtered off, and the liquid partly evaporated. On acidification with HCl a white product separates slowly. The disaccharin can be separated from the unchanged diamide by recrystallisation from water. M.p. ± 320°. The yield is satisfactory. Also the analysis points to a disaccharin. The taste is bitter.

The second disulphonic acid which must yield a disaccharin, is evidently, not formed on direct sulphonation of p-xylene.

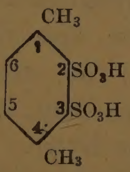
As starting-point for the synthesis of this compound I have chosen 

which is formed on strong nitration of *p*-xylene by the side of 2,5-dinitro-1,4-xylene. The complete separation of these isomers cannot be accomplished, because of the formation of a compound. Small quantities can, however, be separated.

On partial reduction with $(NH_4)_2S$ two nitro-xylidines are formed, one of which is very sparingly soluble in acid. This indicates the structure 

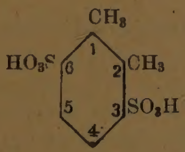
, as in this compound the amido-group is very greatly under the influence of the nitro-group.

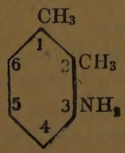
Actually the pure 2,3-dinitro-compound only yields xylidine, which is not easily soluble in acid.

I hope that I shall succeed in preparing the acid 

by substitution from this compound.

The disaccharin from o-xylene.

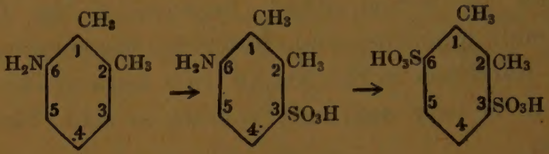
The required sulphonic acid is 

The xylidine 

of Poulenc frères was taken as starting-point.

It is sulphonised with fuming sulphuric acid. First the sulphuric acid salt is formed, which slowly dissolves in the acid. The solution is poured on water and ice. A thick pulpy mass is formed. The precipitate is filtered off and washed, till all the sulphuric acid has been removed. The amido sulphonic acid thus obtained is converted to the Na-salt; from the mother liquor still more of this salt can be obtained by the leadmethod. It is easily soluble and crystallizes in leaflets. By diazotation with Na_2S_2 sulphur is introduced. Already at —5° a violent generation of nitrogen sets in. The Na-salt of the sulphonic acid and the sulpho chloride are obtained in the usual way. The latter is liquid.

The amido group being strongly para-directing, the above conversions may be represented thus:



I hope to prove the structure further by melting with KOH, the corresponding phenol (m.p. 221°) being known.

A satisfactory yield of diamide can only be obtained from disulpho chloride when it is dissolved in benzene, and gaseous NH_3 is introduced. The diamide can be recrystallized from alcohol. Mp. 251°.

On oxidation with $KMnO_4$ I obtained, after the liquid was acidified, a first crystallisation that melts at 275°—280°. This substance is probably the disaccharin. A preliminary micro-analysis is in agreement with this assumption. By means of a mixing-experiment I have shown that diamide and so-called disaccharin are different. The taste is bitter.

Amsterdam
The Hague, March 1924.